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IX. *The Differential Invariants of a Surface, and their Geometric Significance.*

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THE present memoir is devoted to the consideration of the differential invariants of a surface; and these are defined as the functions of the fundamental magnitudes of the surface and of quantities connected with curves upon the surface which remain unchanged in value through all changes of the variables of position on the surface. The idea of differential parameters for relations of space appears to have been introduced by LAMÉ; it is to BELTRAMI* that the earliest investigations of the corresponding quantities in the theory of surfaces are due, as well as many detailed results.†

It is natural to expect that these differential invariants would belong to the general class of differential invariants which constitute LIE's important generalisation of the original theory of invariants and covariants of homogeneous forms. This association has been effected‡ for some classes of differential invariants by Professor ŽORAWSKI, and he has obtained the explicit expression of several of the individual functions.

Professor ŽORAWSKI's method is used in the present memoir. In applying it, a considerable simplification proves to be possible; for it appears that, at a certain stage in the solution of the partial differential equations characteristic of the invariance, the equations which then remain unsolved can be transformed so that they become the partial differential equations of the system of concomitants of a set of simultaneous binary forms. The known results of the latter theory can therefore be used to complete the solution of the partial differential equations, and the result gives the algebraic aggregate of the differential invariants.

This memoir consists of two parts. In the first, the investigation just indicated is carried out; and the explicit expressions of the members of an aggregate, algebraically

* In his memoir, "Sulla teorica generale dei parametri differenziali," 'Mem. Acc. Bologna,' 2nd Series, vol. 8 (1869), pp. 549–590, BELTRAMI gives a sketch of the early history of the subject.

† An account of the theory, developed on the basis of BELTRAMI'S researches, is given by DARBOUX, 'Théorie générale des surfaces,' vol. 3, pp. 193–217; he also gives references to BONNET and LAGUERRE.

‡ In a memoir hereafter quoted (§ 1).

complete up to a certain order, are obtained. In the second part, the geometric significance of the different invariants is the goal; in attaining it, some modifications are made in the aggregate, but they leave it algebraically complete.

The investigation reveals new relations among the intrinsic geometric properties of a curve upon a surface. To the order considered, four such relations exist; and their explicit expressions have been constructed.

PART I.

Construction of the Invariants.

1. In an interesting memoir* published in the 'Acta Mathematica,' Professor Žorawski has developed a method, outlined by Lie,† and has applied it to the determination of certain properties of functions which appertain to a surface and are invariantive, alike under any transformation of the two independent variables and under any deformation of the surface that involves neither tearing nor stretching. In particular, he obtains the number of these functions of any order which are algebraically independent of one another; he also obtains expressions for several functions of the lowest orders belonging to recognised types.

The method, and much of Professor Žorawski's analysis, can be applied to obtain the more extensive class of all the differential functions which, appertaining to a surface and to any set of curves upon the surface, are invariantive under any transformation of the two independent variables. The process, which involves the solution of complete Jacobian systems of the first order and the first degree, only gives the invariantive functions which are algebraically independent of one another; it is not adapted to the construction of the asyzygetic aggregate. Moreover, only some of these functions are invariantive when the surface is deformed without tearing or stretching; they can be selected by inspection, on using the fundamental theorem connected with the theory of the deformation of surfaces.

As far as possible, the notation adopted by Professor Žorawski is used. The analysis, preliminary to the construction of the differential equations which are characteristic of the invariance, is set out briefly; it is needed to make the process intelligible. There is some difference from Professor Žorawski's analysis, mainly (but not entirely) because a beginning is made from the consideration of relative invariants and not of absolute invariants.

2. The independent variables of position on the surface are taken to be x and y. A function f of these variables and of the derivatives of any number of functions

* "Ueber Biegungsinvarianten: eine Anwendung der Lie'schen Gruppentheorie," 'Acta Math.,' vol. 16 (1892–93), pp. 1–64.

† 'Math. Ann.,' vol. 24 (1884), pp. 574, 575.

which involve the invariables is said to be a relative invariant when, if the same function F of new independent variables X and Y and of corresponding new derivatives of the transformed functions be constructed, the relation

$$f = \Omega^{\mu} F$$

is satisfied, where

$$\Omega = \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}.$$

The invariants actually considered are rational, so that μ is an integer. The invariant is said to be absolute where $\mu = 0$.

Now it is known, by LIE'S theory, that the property of invariance will be established if it is possessed for the most general infinitesimal transformation of x and y; accordingly, we shall take

$$X = x + \xi(x, y)\,dt, \quad Y = y + \eta(x, y)\,dt,$$

where ξ and η are arbitrary integral functions of x and y. Derivatives with regard to x and y are required; we write

$$u_{mn} = \frac{\partial^{m+n} u}{\partial x^m \partial y^n},$$

for all values of m and n. Thus, as only the first power of dt is retained, we have

$$\Omega = 1 + (\xi_{10} + \eta_{01})\,dt.$$

The possible Arguments in the Invariants.

3. Next, we have to consider the possible arguments of a differential invariant of a surface. Broadly speaking, these may belong to one or other of three classes:—

(i) the fundamental magnitudes associated with the surface, and their derivatives of any order with respect to x and y;

(ii) functions $\phi(x, y)$, $\psi(x, y)$, ... and their derivatives of any order with respect to x and y;

(iii) the variables x and y, and the derivatives of y of any order with regard to x.

We consider them briefly in turn.

4. Firstly, as regards the fundamental magnitudes: by a known theorem, a surface is defined uniquely (save only as to position and orientation) by the three magnitudes of the first order, usually denoted by E, F, G, and the three magnitudes of the second order, denoted by L, M, N. (If only E, F, G be given, the surface is defined as above, subject also to any deformation that does not involve tearing or stretching.)

These six quantities can occur in the invariantive function required, as well as their derivatives of any order with respect to x and y.

But there is a difficulty as regards the derivatives of L, M, N; for there are two relations, commonly known as the MAINARDI-CODAZZI equations, which express

$$\frac{\partial \mathrm{L}}{\partial y} - \frac{\partial \mathrm{M}}{\partial x}, \quad \frac{\partial \mathrm{M}}{\partial y} - \frac{\partial \mathrm{N}}{\partial x}$$

in terms of L, M, N, E, F, G, and the first derivatives of E, F, G. To avoid this difficulty, it is convenient to introduce the four fundamental magnitudes of the third order, denoted by P, Q, R, S; the six first derivatives of L, M, N can be expressed in terms of P, Q, R, S linearly, together with additive combinations of L, M, N and of the first derivatives of E, F, G.

The second derivatives of L, M, N will thus be expressible in terms of the first derivatives of P, Q, R, S, together with the appropriate additive combinations free from those derivatives. But again there is a difficulty as regards these; for there are three relations, which express

$$\frac{\partial \mathrm{Q}}{\partial x} - \frac{\partial \mathrm{P}}{\partial y}, \quad \frac{\partial \mathrm{R}}{\partial x} - \frac{\partial \mathrm{Q}}{\partial y}, \quad \frac{\partial \mathrm{S}}{\partial x} - \frac{\partial \mathrm{R}}{\partial y}$$

in terms of P, Q, R, S, L, M, N, E, F, G, and the first derivatives of E, F, G. To avoid this new difficulty, it is convenient to introduce the five fundamental magnitudes of the fourth order, denoted by α, β, γ, δ, ϵ; the first derivatives of P, Q, R, S (and therefore the second derivatives of L, M, N) can be expressed linearly in terms of α, β, γ, δ, ϵ, together with additive combinations of P, Q, R, S, L, M, N, E, F, G, and the first derivatives of E, F, G.

And so on, for the derivatives of successive orders of L, M, N; we avoid the difficulty of linear relations among them by the introduction of the successive fundamental magnitudes. The analytical definition* of these magnitudes can be taken in the form

$$ds^2 = \mathrm{E}\,dx^2 + 2\mathrm{F}\,dx\,dy + \mathrm{G}\,dy^2,$$

$$\frac{1}{\rho} = \mathrm{L}\left(\frac{dx}{ds}\right)^2 + 2\mathrm{M}\frac{dx}{ds}\frac{dy}{ds} + \mathrm{N}\left(\frac{dy}{ds}\right)^2,$$

$$= \left(\mathrm{L, M, N}\right)\!\!\left(\frac{dx}{ds}, \frac{dy}{ds}\right)^2,$$

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = \left(\mathrm{P, Q, R, S}\right)\!\!\left(\frac{dx}{ds}, \frac{dy}{ds}\right)^3,$$

$$\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) = \left(\alpha, \beta, \gamma, \delta, \epsilon\right)\!\!\left(\frac{dx}{ds}, \frac{dy}{ds}\right)^4,$$

where ρ is the radius of curvature of the normal section of the surface through the

* See a paper by the author, 'Messenger of Mathematics,' vol. 32 (1903), pp. 68 *et seq.*; see also § 31, *post*.

tangent-line defined by $dx : dy$, and the arc derivatives are effected along the geodesic tangent.*

Accordingly, the quantities of the class under consideration that may occur are E, F, G and their derivatives up to any order, together with the fundamental magnitudes of any order above the first, but without any derivatives of these fundamental magnitudes.†

5. Secondly, as regards functions $\phi(x, y)$, $\psi(x, y)$, . . . and their derivatives: we do not retain the functions themselves, but only their derivatives, for the following reason. The invariantive property is usually some intrinsic geometric property connected with a curve on the surface represented by $\phi =$ constant or zero, $\psi =$ constant or zero, and the like. Accordingly, we retain only derivatives of these functions up to any order; the equations of transformation will show the connection of the order of these derivatives with the order of the derivatives of E, F, G retained.

6. Thirdly, as regards x, y, and the derivatives of y with respect to x up to any order: it is clear that x and y will not occur explicitly, for their presence cannot contribute any element to the factor Ω; it is also clear that they will not occur explicitly, for the further reason that their increments involve ξ and η but not derivatives of ξ or η, whereas all other increments involve derivatives of ξ or η, but neither ξ nor η themselves. Further, after the retention of quantities of the second class, we shall not retain y'. For let the value of y' belong to a curve $\psi = 0$ on the surface, so that

$$\psi_{10} + y' \psi_{01} = 0.$$

We know that

$$\frac{E\psi_{01}^2 - 2F\psi_{01}\psi_{10} + G\psi_{10}^2}{EG - F^2} = I,$$

where I is an absolute invariant; if then we have a differential invariant involving y', we turn it into one involving ψ_{10} and ψ_{01}, by writing

$$y' = -\frac{\psi_{10}}{\psi_{01}};$$

while if we have one involving ψ_{10} and ψ_{01}, we turn it into one involving y', by writing

$$\frac{\psi_{01}}{I} = \frac{\psi_{10}}{y'} = \left\{I\frac{EG - F^2}{E + 2Fy' + Gy'^2}\right\}^{\frac{1}{2}}.$$

It would therefore be unnecessary to retain y', when we retain first derivatives of any number of functions in an earlier class.

Similarly, it can be shown to be unnecessary to retain y'', when we retain second derivatives of any number of functions in an earlier class; and so for other derivatives of y with respect to x.

* See § 31, *post.*

† It will appear that the introduction of these magnitudes not merely avoids the difficulty as regards the derivatives of L, M, N, but also secures a substantial simplification of the expressions of the differential invariants.

Hence we retain none of the third class of possible magnitudes. But after the reasons adduced, we should only be justified in dropping y' from the set of magnitudes when it was otherwise required, if we associated the first derivatives of the appropriate function ψ with the functions already retained; or in dropping y'', if we associated the second derivatives of ψ with the functions already retained; and so for the other derivatives of y. (An example occurs later in § 24.)

NOTE.—In calculations subsidiary to the determination of the geometric significance, it is found necessary to use the relations involving the derivatives of L, M, N, P, Q, R, S; it may therefore be convenient to give their explicit expressions.* They are:—

$$\left.\begin{aligned} P &= L_{10} - 2(L\Gamma + M\Delta) \\ Q &= L_{01} - 2(L\Gamma' + M\Delta') \\ &= M_{10} - (L\Gamma' + M\Delta') - (M\Gamma + N\Delta) \\ R &= M_{01} - (L\Gamma'' + M\Delta'') - (M\Gamma' + N\Delta') \\ &= N_{10} - 2(M\Gamma' + N\Delta') \\ S &= N_{01} - 2(M\Gamma'' + N\Delta'') \end{aligned}\right\},$$

where

$$\left.\begin{aligned} 2V^2\Gamma &= GE_{10} - F(2F_{10} - E_{01}) \\ 2V^2\Gamma' &= GE_{01} - FG_{10} \\ 2V^2\Gamma'' &= G(2F_{01} - G_{10}) - FG_{01} \end{aligned}\right\}, \qquad \left.\begin{aligned} 2V^2\Delta &= E(2F_{10} - E_{01}) - FE_{10} \\ 2V^2\Delta' &= EG_{10} - FE_{01} \\ 2V^2\Delta'' &= EG_{01} - F(2F_{01} - G_{10}) \end{aligned}\right\};$$

and

$$\left.\begin{aligned} \alpha &= P_{10} - 3(P\Gamma + Q\Delta) \\ \beta &= P_{01} - 3(P\Gamma' + Q\Delta') - \tfrac{3}{2}\frac{T^2}{V^2}(FL - EM) \\ &= Q_{10} - (P\Gamma' + Q\Delta') - 2(Q\Gamma + R\Delta) + \tfrac{1}{2}\frac{T^2}{V^2}(FL - EM) \\ \gamma &= Q_{01} - (P\Gamma'' + Q\Delta'') - 2(Q\Gamma' + R\Delta') - \tfrac{1}{2}\frac{T^2}{V^2}(GL - EN) \\ &= R_{10} - 2(Q\Gamma' + R\Delta') - (R\Gamma + S\Delta) + \tfrac{1}{2}\frac{T^2}{V^2}(GL - EN) \\ \delta &= R_{01} - 2(Q\Gamma'' + R\Delta'') - (R\Gamma' + S\Delta') - \tfrac{1}{2}\frac{T^2}{V^2}(GM - FN) \\ &= S_{10} - 3(R\Gamma' + S\Delta') + \tfrac{3}{2}\frac{T^2}{V^2}(GM - FN) \\ \epsilon &= S_{01} - 3(R\Gamma'' + S\Delta'') \end{aligned}\right\},$$

where $T^2 = LN - M^2$.

* They are quoted from the author's paper, mentioned in § 4.

Increments of the Arguments.

7. We now require the increments of the various arguments, corresponding to the increments of x and y. We denote by E′, F′, . . . the same functions of X and Y as E, F, . . . are of x and y; thus, if dE be the increment of E, we have

$$E' = E + dE;$$

and so for the other magnitudes.

Since the relation

$$E\,dx^2 + 2F\,dx\,dy + G\,dy^2 = E'\,dX^2 + 2F'\,dX\,dY + G'\,dY^2$$

holds for all values of dx and dy, we have

$$\begin{aligned} E &= E'\left(\frac{\partial X}{\partial x}\right)^2 + 2F'\frac{\partial X}{\partial x}\frac{\partial Y}{\partial x} + G'\left(\frac{\partial Y}{\partial x}\right)^2 \\ &= E'(1 + 2\xi_{10}\,dt) + 2F'\eta_{10}\,dt, \\ F &= E'\frac{\partial X}{\partial x}\frac{\partial X}{\partial y} + F'\left(\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} + \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}\right) + G\frac{\partial Y}{\partial x}\frac{\partial Y}{\partial y} \\ &= E'\xi_{01}\,dt + F'(1 + \xi_{10}\,dt + \eta_{01}\,dt) + G'\eta_{10}\,dt, \\ G &= E'\left(\frac{\partial X}{\partial y}\right)^2 + 2F'\frac{\partial X}{\partial y}\frac{\partial Y}{\partial y} + G'\left(\frac{\partial Y}{\partial y}\right)^2 \\ &= 2F'\xi_{01}\,dt + G'(1 + 2\eta_{01}\,dt). \end{aligned}$$

We thus have

$$-\,dE = (2E'\xi_{10} + 2F'\eta_{10})\,dt.$$

Now, the differences between E and E′, F and F′, are small quantities of the order dt; hence, when we are retaining only small quantities of the order dt on the right hand side, we can replace E′, F′, G′ by E, F, G respectively; and we find

$$\left.\begin{aligned} -\frac{dE}{dt} &= 2E\xi_{10} + 2F\eta_{10} \\ -\frac{dF}{dt} &= E\xi_{01} + F\xi_{10} + F\eta_{01} + G\eta_{10} \\ -\frac{dG}{dt} &= 2F\xi_{01} + 2G\eta_{01} \end{aligned}\right\}.$$

Similarly, the relation

$$L\,dx^2 + 2M\,dx\,dy + N\,dy^2 = \frac{ds^2}{\rho} = L'\,dX^2 + 2M'\,dX\,dY + N'\,dY^2$$

holds for all values of dx and dy; so that the laws of transformation for L, M, N are the same as for E, F, G. Hence

$$\left.\begin{aligned}
-\frac{d\text{L}}{dt} &= 2\text{L}\xi_{10} + 2\text{M}\eta_{10} \\
-\frac{d\text{M}}{dt} &= \text{L}\xi_{01} + \text{M}\xi_{10} + \text{M}\eta_{01} + \text{N}\eta_{10} \\
-\frac{d\text{N}}{dt} &= 2\text{M}\xi_{01} + 2\text{N}\eta_{01}
\end{aligned}\right\}.$$

Using the relation

$$(\text{P, Q, R, S}\between dx, dy)^3 = ds^3 \cdot \frac{d}{ds}\left(\frac{1}{\rho}\right) = (\text{P}', \text{Q}', \text{R}', \text{S}'\between d\text{X}, d\text{Y})^3$$

in the same way, we find

$$\left.\begin{aligned}
-\frac{d\text{P}}{dt} &= 3\text{P}\xi_{10} + 3\text{Q}\eta_{10} \\
-\frac{d\text{Q}}{dt} &= \text{P}\xi_{01} + 2\text{Q}\xi_{10} + \text{Q}\eta_{01} + 2\text{R}\eta_{10} \\
-\frac{d\text{R}}{dt} &= 2\text{Q}\xi_{01} + \text{R}\xi_{10} + 2\text{R}\eta_{01} + \text{S}\eta_{10} \\
-\frac{d\text{S}}{dt} &= 3\text{R}\xi_{01} + 3\text{S}\eta_{01}
\end{aligned}\right\}.$$

Using the relation

$$(\alpha, \beta, \gamma, \delta, \epsilon\between dx, dy)^4 = ds^4 \cdot \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) = (\alpha', \beta', \gamma', \delta', \epsilon'\between d\text{X}, d\text{Y})^4$$

similarly, we find

$$\left.\begin{aligned}
-\frac{d\alpha}{dt} &= +4\alpha\xi_{10} + 4\beta\eta_{10} \\
-\frac{d\beta}{dt} &= \alpha\xi_{01} + 3\beta\xi_{10} + \beta\eta_{01} + 3\gamma\eta_{10} \\
-\frac{d\gamma}{dt} &= 2\beta\xi_{01} + 2\gamma\xi_{10} + 2\gamma\eta_{01} + 2\delta\eta_{10} \\
-\frac{d\delta}{dt} &= 3\gamma\xi_{01} + \delta\xi_{10} + 3\delta\eta_{01} + \epsilon\eta_{10} \\
-\frac{d\epsilon}{dt} &= 4\delta\xi_{01} + 4\epsilon\eta_{01}
\end{aligned}\right\}.$$

And so for the increments of the other fundamental magnitudes.

8. The increments of the derivatives of E, F, G are required; they can be obtained by the following method, differing from that which is adopted by Professor Žorawski. Let x and y become $x + h$ and $y + k$ respectively, and let the consequent new values of X and Y be X + H, Y + K; then

$$H = (X + H) - H = h + \{\xi(x + h, y + k) - \xi(x, y)\}\, dt = h + A\, dt,$$

where

$$A = \sum_{r=0} \sum_{s=0}' \xi_{rs} \frac{h^r k^s}{r\,!\, s\,!},$$

and Σ' implies that r and s may not be zero together.

Similarly

$$K = k + B\, dt,$$

where

$$B = \sum_{r=0} \sum_{s=0}' \eta_{rs} \frac{h^r k^s}{r\,!\, s\,!}$$

with the same signification for Σ' as before; and thus, for all values of p and q, we have

$$H^p K^q = h^p k^q + (p h^{p-1} k^q A + q h^p k^{q-1} B)\, dt.$$

Now, as the relation

$$E = E'(1 + 2\xi_{10}\, dt) + 2F'\eta_{10}\, dt$$

holds for all values of x and y, it follows that

$$E(x + h, y + k) = E(X + H, Y + K)\{1 + 2\xi_{10}(x + h, y + k)\, dt\}$$
$$+ 2F(X + H, Y + K)\,\eta_{10}(x + h, y + k)\, dt.$$

Let both sides be expanded in powers of h and k; then $\frac{E_{mn}}{m\,!\, n\,!}$ = coefficient of $h^m k^n$ in the expansion of

$$\left[\sum_{p=0} \sum_{q=0} \frac{E'_{pq}}{p\,!\, q\,!} \{h^p k^q + (p h^{p-1} k^q A + q h^p k^{q-1} B)\, dt\}\right] \left[1 + 2 \sum_{r=0} \sum_{s=0} \xi_{r+1,s} \frac{h^r k^s}{r\,!\, s\,!}\, dt\right]$$
$$+ 2\left[\sum_{p=0} \sum_{q=0} \frac{F'_{pq}}{p\,!\, q\,!} \{h^p k^q + (p h^{p-1} k^q A + q h^p k^{q-1} B)\, dt\}\right] \left[\sum_{r=0} \sum_{s=0} \eta_{r+1,s} \frac{h^r k^s}{r\,!\, s\,!}\, dt\right].$$

Remembering that the first power of dt alone is to be retained, we find this coefficient to be

$$\frac{E'_{mn}}{m\,!\, n\,!}$$
$$+ \Sigma\Sigma' \frac{1}{(m - r + 1)\,!\,(n - s)\,!\, r\,!\, s\,!} (m - r + 1)\,\xi_{rs} E'_{m-r+1,\, n-s}\, dt$$
$$+ 2\Sigma\Sigma \frac{1}{(m - r)\,!\,(n - s)\,!\, r\,!\, s\,!} \xi_{r+1,s} E'_{m-r,\, n-s}\, dt$$
$$+ \Sigma\Sigma' \frac{1}{(m - r)\,!\,(n - s + 1)\,!\, r\,!\, s\,!} (n - s + 1)\,\eta_{rs} E'_{m-r,\, n-s+1}\, dt$$
$$+ 2\Sigma\Sigma \frac{1}{(m - r)\,!\,(n - s)\,!\, r\,!\, s\,!} \eta_{r+1,s} F'_{m-r,\, n-s}\, dt;$$

the first summation $\Sigma\Sigma'$ does not occur if $r = m + 1$, the second summation $\Sigma\Sigma'$ does not occur if $s = n + 1$, and in neither of them may r and s vanish together. Writing

$$\binom{m}{r} = \frac{m\,!}{(m-r)\,!\,r\,!}, \qquad \binom{n}{s} = \frac{n\,!}{(n-s)\,!\,s\,!},$$

$$E'_{mn} = E_{mn} + dE_{mn},$$

we have

$$
\begin{aligned}
-\frac{dE_{mn}}{dt} = & \sum_{r=0}^{m}\sum_{s=0}^{n}{}' \binom{m}{r}\binom{n}{s}\xi_{rs}E'_{m-r+1,\,n-s} \\
& + 2\sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\xi_{r+1,\,s}E'_{m-r,\,n-s} \\
& + \sum_{r=0}^{m}\sum_{s=0}^{n}{}' \binom{m}{r}\binom{n}{s}\eta_{rs}E'_{m-r,\,n-s+1} \\
& + 2\sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\eta_{r+1,\,s}F'_{m-r,\,n-s}.
\end{aligned}
$$

Proceeding similarly from the expressions for F and G, we find

$$
\begin{aligned}
-\frac{dF_{mn}}{dt} = & \sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\xi_{r,\,s+1}E'_{m-r,\,n-s} \\
& + \sum_{r=0}^{m}\sum_{s=0}^{n}{}' \binom{m}{r}\binom{n}{s}\xi_{rs}F'_{m-r+1,\,n-s} \\
& + \sum_{r=0}^{m}\sum_{s=0}^{n}{}' \binom{m}{r}\binom{n}{s}\eta_{rs}F'_{m-r,\,n-s+1} \\
& + \sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}(\xi_{r+1,\,s} + \eta_{r,\,s+1})\,F'_{m-r,\,n-s} \\
& + \sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\eta_{r+1,\,s}G'_{m-r,\,n-s},
\end{aligned}
$$

and

$$
\begin{aligned}
-\frac{dG_{mn}}{dt} = & 2\sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\xi_{r,\,s+1}F'_{m-r,\,n-s} \\
& + 2\sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\eta_{r,\,s+1}G'_{m-r,\,n-s} \\
& + \sum_{r=0}^{m}\sum_{s=0}^{n}{}' \binom{m}{r}\binom{n}{s}\xi_{rs}G'_{m-r+1,\,n-s} \\
& + \sum_{r=0}^{m}\sum_{s=0}^{n}{}' \binom{m}{r}\binom{n}{s}\eta_{rs}G'_{m-r,\,n-s+1}.
\end{aligned}
$$

NOTE.—As we now have the first increment of the quantities E_{mn}, F_{mn}, G_{mn}, and as the second increments are not required, the quantities E′, F′, G′ on the right-hand sides can be replaced by E, F, G, without affecting the values of the first increments.

9. In particular, we have

$$\left.\begin{aligned}
-\frac{dE_{10}}{dt} &= 3\xi_{10}E_{10} + 2\xi_{20}E + \eta_{10}(E_{01} + 2F_{10}) + 2\eta_{20}F \\
-\frac{dE_{01}}{dt} &= 2\xi_{10}E_{01} + \xi_{01}E_{10} + 2\xi_{11}E + \eta_{01}E_{01} + 2\eta_{10}F_{01} + 2\eta_{11}F \\
-\frac{dE_{20}}{dt} &= 4\xi_{10}E_{20} + 5\xi_{20}E_{10} + 2\xi_{30}E \\
&\quad + 2\eta_{10}(E_{11} + F_{20}) + \eta_{20}(E_{01} + 4F_{10}) + 2F\eta_{30} \\
-\frac{dE_{11}}{dt} &= 3\xi_{10}E_{11} + \xi_{01}E_{20} + 2\xi_{20}E_{01} + 3\xi_{11}E_{10} + 2\xi_{21}E \\
&\quad + \eta_{10}(E_{02} + 2F_{11}) + \eta_{01}E_{11} + 2\eta_{20}F_{01} + \eta_{11}(E_{01} + 2F_{10}) + 2\eta_{21}F \\
-\frac{dE_{02}}{dt} &= 2\xi_{10}E_{02} + 2\xi_{01}E_{11} + 4\xi_{11}E_{01} + \xi_{02}E_{10} + 2\xi_{12}E \\
&\quad + 2\eta_{01}E_{02} + 2\eta_{10}F_{02} + 4\eta_{11}F_{01} + \eta_{02}E_{01} + 2\eta_{12}F
\end{aligned}\right\};$$

$$\left.\begin{aligned}
-\frac{dF_{10}}{dt} &= 2\xi_{10}F_{10} + \xi_{01}E_{10} + \xi_{20}F + \xi_{11}E \\
&\quad + \eta_{10}(F_{01} + G_{10}) + \eta_{01}F_{10} + \eta_{20}G + \eta_{11}F \\
-\frac{dF_{01}}{dt} &= \xi_{10}F_{01} + \xi_{01}(E_{01} + F_{10}) + \xi_{11}F + \xi_{02}E \\
&\quad + \eta_{10}G_{01} + 2\eta_{01}F_{01} + \eta_{11}G + \eta_{02}F \\
-\frac{dF_{20}}{dt} &= 3\xi_{10}F_{20} + \xi_{01}E_{20} + 3\xi_{20}F_{10} + 2\xi_{11}E_{10} + \xi_{21}E + \xi_{30}F \\
&\quad + \eta_{01}F_{20} + 2\eta_{10}F_{11} + \eta_{10}G_{20} + \eta_{20}(2G_{10} + F_{01}) + 2\eta_{11}F_{10} + \eta_{30}G + \eta_{21}F \\
-\frac{dF_{11}}{dt} &= 2\xi_{10}F_{11} + \xi_{01}(E_{11} + F_{20}) + \xi_{20}F_{01} + \xi_{11}(E_{01} + 2F_{10}) + \xi_{02}E_{10} + \xi_{21}F + \xi_{12}E \\
&\quad + 2\eta_{01}F_{11} + \eta_{10}(F_{02} + G_{11}) + \eta_{20}G_{01} + \eta_{11}(2F_{01} + G_{10}) + \eta_{02}F_{10} + \eta_{21}G + \eta_{12}F \\
-\frac{dF_{02}}{dt} &= \xi_{10}F_{02} + \xi_{01}(E_{02} + 2F_{11}) + 2\xi_{11}F_{01} + \xi_{02}(2E_{01} + F_{10}) + \xi_{12}F + \xi_{03}E \\
&\quad + 3\eta_{01}F_{02} + \eta_{10}G_{02} + 2\eta_{11}G_{01} + 3\eta_{02}F_{01} + \eta_{12}G + \eta_{03}F
\end{aligned}\right\};$$

$$\left.\begin{aligned}
-\frac{dG_{10}}{dt} &= \xi_{10}G_{10} + 2\xi_{01}F_{10} + 2\xi_{11}F + \eta_{10}G_{01} + 2\eta_{01}G_{10} + 2\eta_{11}G \\
-\frac{dG_{01}}{dt} &= \xi_{01}(G_{10} + 2F_{01}) + 2\xi_{02}F + 3\eta_{01}G_{01} + 2\eta_{02}G \\
-\frac{dG_{20}}{dt} &= 2\xi_{10}G_{20} + 2\xi_{01}F_{20} + \xi_{20}G_{10} + 4\xi_{11}F_{10} + 2\xi_{21}F \\
&\quad + 2\eta_{01}G_{20} + 2\eta_{10}G_{11} + \eta_{20}G_{01} + 4\eta_{11}G_{10} + 2\eta_{21}G \\
-\frac{dG_{11}}{dt} &= \xi_{10}G_{11} + \xi_{01}(G_{20} + 2F_{11}) + \xi_{11}(2F_{01} + G_{10}) + 2\xi_{02}F_{10} + 2\xi_{12}F \\
&\quad + 3\eta_{01}G_{11} + \eta_{10}G_{02} + 3\eta_{11}G_{01} + 2\eta_{02}G_{10} + 2\eta_{12}G \\
-\frac{dG_{02}}{dt} &= 2\xi_{01}(F_{02} + G_{11}) + \xi_{02}(G_{10} + 4F_{01}) + 2\xi_{03}F \\
&\quad + 4\eta_{01}G_{02} + 5\eta_{02}G_{01} + 2\eta_{03}G
\end{aligned}\right\}.$$

10. We require expressions for the increments of the derivatives of functions such as $\phi(x, y)$, $\psi(x, y)$, . . . ; for this purpose, we proceed as before. We have

$$\phi(x + h, y + k) = \phi'(X + H, Y + K);$$

and therefore

$$\frac{\phi_{mn}}{m!\,n!} = \text{coefficient of } h^m k^n \text{ in expansion of } \phi(X + H, Y + K)$$

$$= \ldots\ldots\ldots \sum_p \sum_q \frac{\phi'_{pq}}{p!\,q!} H^p K^q$$

$$= \ldots\ldots\ldots \sum_p \sum_q \frac{\phi'_{pq}}{p!\,q!} \{h^p k^q + (ph^{p-1}k^q A + qh^p k^{q-1} B)\, dt\}$$

$$= \frac{\phi'_{mn}}{m!\,n!} + U\, dt,$$

where U is the coefficient of $h^m k^n$ in

$$\sum_p \sum_q \frac{\phi'_{pq}}{p!\,q!} (ph^{p-1}k^q A + qh^p k^{q-1} B),$$

that is, in

$$\sum_p \sum_q \sum_r \sum_s \frac{\phi'_{pq}}{p!\,q!\,r!\,s!} (ph^{r+p-1}k^{q+s}\xi_{rs} + qh^{r+p}k^{q+s-1}\eta_{rs}),$$

where in the summation r and s do not vanish together and, if either p or q be zero, the corresponding term ceases to occur.

Writing

$$\phi'_{mn} = \phi_{mn} + d\phi_{mn},$$

we have

$$-\frac{d\phi_{mn}}{dt} = \sum_{r=0}^{m} \sum_{s=0}^{n}{}' \binom{m}{r}\binom{n}{s} \{\phi'_{m+1-r,\, n-s}\xi_{rs} + \phi'_{m-r,\, n+1-s}\eta_{rs}\},$$

which gives the required increments for derivatives of a function ϕ. Similarly of course for the increments of the derivatives of all functions similar to ϕ.

NOTE.—Just as in the expressions for the increments of the various derivatives of E, F, G, we can replace, in the expressions for the increments of the various derivatives of a function ϕ, the various quantities $\phi'_{\mu\nu}$ on the right-hand sides by $\phi_{\mu\nu}$ without affecting the values of the first increments. As before, second increments are not needed for our purpose.

11. In particular, we have

$$\left.\begin{aligned} -\frac{d\phi_{10}}{dt} &= \phi_{10}\xi_{10} + \phi_{01}\eta_{10} \\ -\frac{d\phi_{01}}{dt} &= \phi_{10}\xi_{01} + \phi_{01}\eta_{01} \end{aligned}\right\};$$

$$\left.\begin{aligned} -\frac{d\phi_{20}}{dt} &= 2\phi_{20}\xi_{10} + \phi_{10}\xi_{20} + 2\phi_{11}\eta_{10} + \phi_{01}\eta_{20} \\ -\frac{d\phi_{11}}{dt} &= \phi_{11}\xi_{10} + \phi_{20}\xi_{01} + \phi_{10}\xi_{11} + \phi_{02}\eta_{10} + \phi_{11}\eta_{01} + \phi_{01}\eta_{11} \\ -\frac{d\phi_{02}}{dt} &= 2\phi_{11}\xi_{01} + \phi_{10}\xi_{02} + 2\phi_{02}\eta_{01} + \phi_{01}\eta_{02} \end{aligned}\right\};$$

$$\left.\begin{aligned} -\frac{d\phi_{30}}{dt} &= 3\phi_{30}\xi_{10} + 3\phi_{20}\xi_{20} + \phi_{10}\xi_{30} + 3\phi_{21}\eta_{10} + 3\phi_{11}\eta_{20} + \phi_{01}\eta_{30} \\ -\frac{d\phi_{21}}{dt} &= 2\phi_{21}\xi_{10} + \phi_{30}\xi_{01} + \phi_{11}\xi_{20} + 2\phi_{20}\xi_{11} + \phi_{10}\xi_{21} \\ &\quad + 2\phi_{12}\eta_{10} + \phi_{21}\eta_{01} + \phi_{02}\eta_{20} + 2\phi_{11}\eta_{11} + \phi_{01}\eta_{21} \\ -\frac{d\phi_{12}}{dt} &= \phi_{12}\xi_{10} + 2\phi_{21}\xi_{01} + 2\phi_{11}\xi_{11} + \phi_{20}\xi_{02} + \phi_{10}\xi_{12} \\ &\quad + \phi_{03}\eta_{10} + 2\phi_{12}\eta_{01} + 2\phi_{02}\eta_{11} + \phi_{11}\eta_{02} + \phi_{01}\eta_{12} \\ -\frac{d\phi_{03}}{dt} &= 3\phi_{12}\xi_{01} + 3\phi_{11}\xi_{02} + \phi_{10}\xi_{03} + 3\phi_{03}\eta_{01} + 3\phi_{02}\eta_{02} + \phi_{01}\eta_{03} \end{aligned}\right\}.$$

12. A comparison of the expressions of the increments of the derivatives of E, F, G on the one hand, and those of the derivatives of a typical function ϕ on the other, leads to one immediate inference as to the arguments that enter into the composition of a differential invariant. Suppose that such an invariant is required to involve derivatives of a function ϕ up to order M in x and y combined; the increments of these derivatives involve (among others) the quantities

$$\xi_{M0}, \quad \xi_{M1}, \ . \ . \ . \ , \quad \xi_{0M}\,; \quad \eta_{M0}, \quad \eta_{M1}, \ . \ . \ . \ , \quad \eta_{0M}.$$

The invariantive property requires that the terms involving these quantities must (if they do not balance one another) be balanced by other terms involving these same quantities; and therefore derivatives of E, F, G up to order M − 1 in x and y combined must occur. And conversely.

In particular, if derivatives of ϕ of the third order occur in an invariantive function, it must contain derivatives of E, F, G of the second order.

The Differential Equations Defining the Invariants.

13. The invariantive property is used, exactly as in Professor Žorawski's application of Lie's method, to obtain partial differential equations of the first order satisfied by any invariantive function. We proceed from an equation such as

$$f' = \Omega^{-\mu} f\,;$$

we substitute, in each of the arguments such as u', where

$$u' = u + dt \cdot \frac{du}{dt}$$

the proper value of $\frac{du}{dt}$ obtained above for the various arguments; we also write

$$\Omega = 1 + (\xi_{10} + \eta_{01})\,dt;$$

and then, according to LIE's theory, we equate the coefficient of dt on the two sides. The functions ξ and η are arbitrary; and therefore, in this new equation, the coefficients of the various derivatives of ξ and η on the two sides are equal. We thus obtain a number of partial differential equations of the first order satisfied by f. The construction of the form of f depends upon the manipulation of the equations.

14. The whole process will be sufficiently illustrated in its details if we construct the algebraically independent aggregate of differential invariants which involve derivatives of two* functions ϕ and ψ up to the third order inclusive. In order to take full account of the increments of such derivatives, it is desirable and necessary to retain derivatives of E, F, G up to the second order and, in place of the derivatives of L, M, N of that order, to retain the fundamental magnitudes of the second, the third, and the fourth orders. Thus the invariantive function involves some or all of the quantities

$$\mathrm{E},\ \mathrm{E}_{10},\ \mathrm{E}_{01},\ \mathrm{E}_{20},\ \mathrm{E}_{11},\ \mathrm{E}_{02};$$
$$\mathrm{F},\ \mathrm{F}_{10},\ \mathrm{F}_{01},\ \mathrm{F}_{20},\ \mathrm{F}_{11},\ \mathrm{F}_{02};$$
$$\mathrm{G},\ \mathrm{G}_{10},\ \mathrm{G}_{01},\ \mathrm{G}_{20},\ \mathrm{G}_{11},\ \mathrm{G}_{02};$$
$$\mathrm{L},\ \mathrm{M},\ \mathrm{N};$$
$$\mathrm{P},\ \mathrm{Q},\ \mathrm{R},\ \mathrm{S};$$
$$\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon;$$
$$\phi_{10},\ \phi_{01},\ \phi_{20},\ \phi_{11},\ \phi_{02},\ \phi_{30},\ \phi_{21},\ \phi_{12},\ \phi_{03};$$
$$\psi_{10},\ \psi_{01},\ \psi_{20},\ \psi_{11},\ \psi_{02},\ \psi_{30},\ \psi_{21},\ \psi_{12},\ \psi_{03}.$$

Denoting any one of these arguments by u, the invariantive property gives

$$f(\ldots, u', \ldots) = \Omega^{-\mu} f(\ldots, u, \ldots),$$

that is,

$$f\left(\ldots, u + \frac{du}{dt}\,dt, \ldots\right) = \{1 + (\xi_{10} + \eta_{10})\,dt\}^{-\mu} f(\ldots, u, \ldots),$$

and therefore

$$\sum_u \frac{\partial f}{\partial u}\frac{du}{dt} = -\mu\,(\xi_{10} + \eta_{10})f.$$

* The form of the results indicates the form of the results when more than two functions occur. Moreover, if more than two functions of the type of ϕ and ψ be considered, they are connected by an identical relation.

Substituting for $\frac{du}{dt}$ the respective values for the respective arguments, and equating the coefficients of the various derivatives of ξ and η, we have the requisite partial differential equations. They are :—

$$\begin{aligned}\mu f = {} & 2E\frac{\partial f}{\partial E} + F\frac{\partial f}{\partial F} + 2L\frac{\partial f}{\partial L} + M\frac{\partial f}{\partial M} + 3P\frac{\partial f}{\partial P} + 2Q\frac{\partial f}{\partial Q} + R\frac{\partial f}{\partial R} \\ & + 4\alpha\frac{\partial f}{\partial \alpha} + 3\beta\frac{\partial f}{\partial \beta} + 2\gamma\frac{\partial f}{\partial \gamma} + \delta\frac{\partial f}{\partial \delta} \\ & + 3E_{10}\frac{\partial f}{\partial E_{10}} + 2E_{01}\frac{\partial f}{\partial E_{01}} + 4E_{20}\frac{\partial f}{\partial E_{20}} + 3E_{11}\frac{\partial f}{\partial E_{11}} + 2E_{02}\frac{\partial f}{\partial E_{02}} \\ & + 2F_{10}\frac{\partial f}{\partial F_{10}} + F_{01}\frac{\partial f}{\partial F_{01}} + 3F_{20}\frac{\partial f}{\partial F_{20}} + 2F_{11}\frac{\partial f}{\partial F_{11}} + F_{02}\frac{\partial f}{\partial F_{02}} \\ & + G_{10}\frac{\partial f}{\partial G_{10}} + 2G_{20}\frac{\partial f}{\partial G_{20}} + G_{11}\frac{\partial f}{\partial G_{11}} \\ & + \phi_{10}\frac{\partial f}{\partial \phi_{10}} + 2\phi_{20}\frac{\partial f}{\partial \phi_{20}} + \phi_{11}\frac{\partial f}{\partial \phi_{11}} + 3\phi_{30}\frac{\partial f}{\partial \phi_{30}} + 2\phi_{21}\frac{\partial f}{\partial \phi_{21}} + \phi_{12}\frac{\partial f}{\partial \phi_{12}} \\ & + \psi_{10}\frac{\partial f}{\partial \psi_{10}} + 2\psi_{20}\frac{\partial f}{\partial \psi_{20}} + \psi_{11}\frac{\partial f}{\partial \psi_{11}} + 3\psi_{30}\frac{\partial f}{\partial \psi_{30}} + 2\psi_{21}\frac{\partial f}{\partial \psi_{21}} + \psi_{12}\frac{\partial f}{\partial \psi_{12}} \quad . \quad . \quad . \quad (I_1),\end{aligned}$$

$$\begin{aligned}\mu f = {} & F\frac{\partial f}{\partial F} + 2G\frac{\partial f}{\partial G} + M\frac{\partial f}{\partial M} + 2N\frac{\partial f}{\partial N} + Q\frac{\partial f}{\partial Q} + 2R\frac{\partial f}{\partial R} + 3S\frac{\partial f}{\partial S} \\ & + \beta\frac{\partial f}{\partial \beta} + 2\gamma\frac{\partial f}{\partial \gamma} + 3\delta\frac{\partial f}{\partial \delta} + 4\epsilon\frac{\partial f}{\partial \epsilon} \\ & + E_{01}\frac{\partial f}{\partial E_{01}} + E_{11}\frac{\partial f}{\partial E_{11}} + 2E_{02}\frac{\partial f}{\partial E_{02}} \\ & + F_{10}\frac{\partial f}{\partial F_{10}} + 2F_{01}\frac{\partial f}{\partial F_{01}} + F_{20}\frac{\partial f}{\partial F_{20}} + 2F_{11}\frac{\partial f}{\partial F_{11}} + 3F_{02}\frac{\partial f}{\partial F_{02}} \\ & + 2G_{10}\frac{\partial f}{\partial G_{10}} + 3G_{01}\frac{\partial f}{\partial G_{01}} + 2G_{20}\frac{\partial f}{\partial G_{20}} + 3G_{11}\frac{\partial f}{\partial G_{11}} + 4G_{02}\frac{\partial f}{\partial G_{02}} \\ & + \phi_{01}\frac{\partial f}{\partial \phi_{01}} + \phi_{11}\frac{\partial f}{\partial \phi_{11}} + 2\phi_{02}\frac{\partial f}{\partial \phi_{02}} + \phi_{21}\frac{\partial f}{\partial \phi_{21}} + 2\phi_{12}\frac{\partial f}{\partial \phi_{12}} + 3\phi_{03}\frac{\partial f}{\partial \phi_{03}} \\ & + \psi_{01}\frac{\partial f}{\partial \psi_{01}} + \psi_{11}\frac{\partial f}{\partial \psi_{11}} + 2\psi_{02}\frac{\partial f}{\partial \psi_{02}} + \psi_{21}\frac{\partial f}{\partial \psi_{21}} + 2\psi_{12}\frac{\partial f}{\partial \psi_{12}} + 3\psi_{03}\frac{\partial f}{\partial \psi_{03}} \quad . \quad . \quad (I_2),\end{aligned}$$

which come from the coefficients of ξ_{10}, η_{01} respectively ;

$$\mathrm{E}\frac{\partial f}{\partial \mathrm{F}} + 2\mathrm{F}\frac{\partial f}{\partial \mathrm{G}} + \mathrm{L}\frac{\partial f}{\partial \mathrm{M}} + 2\mathrm{M}\frac{\partial f}{\partial \mathrm{N}} + \mathrm{P}\frac{\partial f}{\partial \mathrm{Q}} + 2\mathrm{Q}\frac{\partial f}{\partial \mathrm{R}} + 3\mathrm{R}\frac{\partial f}{\partial \mathrm{S}}$$
$$+ \alpha\frac{\partial f}{\partial \beta} + 2\beta\frac{\partial f}{\partial \gamma} + 3\gamma\frac{\partial f}{\partial \delta} + 4\delta\frac{\partial f}{\partial \epsilon}$$
$$+ \mathrm{E}_{10}\frac{\partial f}{\partial \mathrm{E}_{01}} + \mathrm{E}_{20}\frac{\partial f}{\partial \mathrm{E}_{11}} + 2\mathrm{E}_{11}\frac{\partial f}{\partial \mathrm{E}_{02}}$$
$$+ \mathrm{E}_{10}\frac{\partial f}{\partial \mathrm{F}_{10}} + (\mathrm{E}_{01} + \mathrm{F}_{10})\frac{\partial f}{\partial \mathrm{F}_{01}} + \mathrm{E}_{20}\frac{\partial f}{\partial \mathrm{F}_{20}} + (\mathrm{E}_{11} + \mathrm{F}_{20})\frac{\partial f}{\partial \mathrm{F}_{11}} + (\mathrm{E}_{02} + 2\mathrm{F}_{11})\frac{\partial f}{\partial \mathrm{F}_{02}}$$
$$+ 2\mathrm{F}_{10}\frac{\partial f}{\partial \mathrm{G}_{10}} + (\mathrm{G}_{10} + 2\mathrm{F}_{01})\frac{\partial f}{\partial \mathrm{G}_{01}} + 2\mathrm{F}_{20}\frac{\partial f}{\partial \mathrm{G}_{20}} + (\mathrm{G}_{20} + 2\mathrm{F}_{11})\frac{\partial f}{\partial \mathrm{G}_{11}} + (2\mathrm{F}_{02} + 2\mathrm{G}_{11})\frac{\partial f}{\partial \mathrm{G}_{02}}$$
$$+ \phi_{10}\frac{\partial f}{\partial \phi_{01}} + \phi_{20}\frac{\partial f}{\partial \phi_{11}} + 2\phi_{11}\frac{\partial f}{\partial \phi_{02}} + \phi_{30}\frac{\partial f}{\partial \phi_{21}} + 2\phi_{21}\frac{\partial f}{\partial \phi_{12}} + 3\phi_{12}\frac{\partial f}{\partial \phi_{03}}$$
$$+ \psi_{10}\frac{\partial f}{\partial \psi_{01}} + \psi_{20}\frac{\partial f}{\partial \psi_{11}} + 2\psi_{11}\frac{\partial f}{\partial \psi_{02}} + \psi_{30}\frac{\partial f}{\partial \psi_{21}} + 2\psi_{21}\frac{\partial f}{\partial \psi_{12}} + 3\psi_{12}\frac{\partial f}{\partial \psi_{03}} = 0 \quad . \quad . \quad (\mathrm{I}_3),$$

$$2\mathrm{F}\frac{\partial f}{\partial \mathrm{E}} + \mathrm{G}\frac{\partial f}{\partial \mathrm{F}} + 2\mathrm{M}\frac{\partial f}{\partial \mathrm{L}} + \mathrm{N}\frac{\partial f}{\partial \mathrm{M}} + 3\mathrm{Q}\frac{\partial f}{\partial \mathrm{P}} + 2\mathrm{R}\frac{\partial f}{\partial \mathrm{Q}} + \mathrm{S}\frac{\partial f}{\partial \mathrm{R}}$$
$$+ 4\beta\frac{\partial f}{\partial \alpha} + 3\gamma\frac{\partial f}{\partial \beta} + 2\delta\frac{\partial f}{\partial \gamma} + \epsilon\frac{\partial f}{\partial \delta}$$
$$+ (\mathrm{E}_{01} + 2\mathrm{F}_{10})\frac{\partial f}{\partial \mathrm{E}_{10}} + 2\mathrm{F}_{01}\frac{\partial f}{\partial \mathrm{E}_{01}} + (2\mathrm{E}_{11} + 2\mathrm{F}_{20})\frac{\partial f}{\partial \mathrm{E}_{20}} + (\mathrm{E}_{02} + 2\mathrm{F}_{11})\frac{\partial f}{\partial \mathrm{E}_{11}} + 2\mathrm{F}_{02}\frac{\partial f}{\partial \mathrm{E}_{02}}$$
$$+ (\mathrm{F}_{01} + \mathrm{G}_{10})\frac{\partial f}{\partial \mathrm{F}_{10}} + \mathrm{G}_{01}\frac{\partial f}{\partial \mathrm{F}_{01}} + (2\mathrm{F}_{11} + \mathrm{G}_{20})\frac{\partial f}{\partial \mathrm{F}_{20}} + (\mathrm{F}_{02} + \mathrm{G}_{11})\frac{\partial f}{\partial \mathrm{F}_{11}} + \mathrm{G}_{02}\frac{\partial f}{\mathrm{F}_{02}}$$
$$+ \mathrm{G}_{01}\frac{\partial f}{\partial \mathrm{G}_{10}} + 2\mathrm{G}_{11}\frac{\partial f}{\partial \mathrm{G}_{20}} + \mathrm{G}_{02}\frac{\partial f}{\partial \mathrm{G}_{11}}$$
$$+ \phi_{01}\frac{\partial f}{\partial \phi_{10}} + 2\phi_{11}\frac{\partial f}{\partial \phi_{20}} + \phi_{02}\frac{\partial f}{\partial \phi_{11}} + 3\phi_{21}\frac{\partial f}{\partial \phi_{30}} + 2\phi_{12}\frac{\partial f}{\partial \phi_{21}} + \phi_{03}\frac{\partial f}{\partial \phi_{12}}$$
$$+ \psi_{01}\frac{\partial f}{\partial \psi_{10}} + 2\psi_{11}\frac{\partial f}{\partial \psi_{20}} + \psi_{02}\frac{\partial f}{\partial \psi_{11}} + 3\psi_{21}\frac{\partial f}{\partial \psi_{30}} + \psi_{12}\frac{\partial f}{\partial \psi_{21}} + \psi_{03}\frac{\partial f}{\partial \psi_{12}} = 0 \quad . \quad (\mathrm{I}_4),$$

which come from the coefficients of ξ_{01}, η_{10} respectively ;

$$2\mathrm{E}\frac{\partial f}{\partial \mathrm{E}_{10}} + 5\mathrm{E}_{10}\frac{\partial f}{\partial \mathrm{E}_{20}} + 2\mathrm{E}_{01}\frac{\partial f}{\partial \mathrm{E}_{11}}$$
$$+ \mathrm{F}\frac{\partial f}{\partial \mathrm{F}_{10}} + 3\mathrm{F}_{10}\frac{\partial f}{\partial \mathrm{F}_{20}} + \mathrm{F}_{01}\frac{\partial f}{\partial \mathrm{F}_{11}}$$
$$+ \mathrm{G}_{10}\frac{\partial f}{\partial \mathrm{G}_{20}}$$
$$+ \phi_{10}\frac{\partial f}{\partial \phi_{20}} + 3\phi_{20}\frac{\partial f}{\partial \phi_{30}} + \phi_{11}\frac{\partial f}{\partial \phi_{21}} + \psi_{10}\frac{\partial f}{\partial \psi_{20}} + 3\psi_{20}\frac{\partial f}{\partial \psi_{30}} + \psi_{11}\frac{\partial f}{\partial \psi_{21}} = 0 \quad . \quad . \quad (\mathrm{II}_1),$$

$$2\mathrm{E}\frac{\partial f}{\partial \mathrm{E}_{01}} + 3\mathrm{E}_{10}\frac{\partial f}{\partial \mathrm{E}_{11}} + 4\mathrm{E}_{01}\frac{\partial f}{\partial \mathrm{E}_{02}}$$
$$+ \mathrm{E}\frac{\partial f}{\partial \mathrm{F}_{10}} + \mathrm{F}\frac{\partial f}{\partial \mathrm{F}_{01}} + 2\mathrm{E}_{10}\frac{\partial f}{\partial \mathrm{F}_{20}} + (\mathrm{E}_{01} + 2\mathrm{F}_{10})\frac{\partial f}{\partial \mathrm{F}_{11}} + 2\mathrm{F}_{01}\frac{\partial f}{\partial \mathrm{F}_{02}}$$
$$+ 2\mathrm{F}\frac{\partial f}{\partial \mathrm{G}_{10}} + 4\mathrm{F}_{10}\frac{\partial f}{\partial \mathrm{G}_{20}} + (2\mathrm{F}_{01} + \mathrm{G}_{10})\frac{\partial f}{\partial \mathrm{G}_{11}}$$
$$+ \phi_{10}\frac{\partial f}{\partial \phi_{11}} + 2\phi_{20}\frac{\partial f}{\partial \phi_{21}} + 2\phi_{11}\frac{\partial f}{\partial \phi_{12}} + \psi_{10}\frac{\partial f}{\partial \psi_{11}} + 2\psi_{20}\frac{\partial f}{\partial \psi_{21}} + 2\psi_{11}\frac{\partial f}{\partial \psi_{12}} = 0 \quad . \quad (\mathrm{II}_2),$$

$$\mathrm{E}_{10}\frac{\partial f}{\partial \mathrm{E}_{02}}$$
$$+ \mathrm{E}\frac{\partial f}{\partial \mathrm{F}_{01}} + \mathrm{E}_{10}\frac{\partial f}{\partial \mathrm{F}_{11}} + (2\mathrm{E}_{01} + \mathrm{F}_{10})\frac{\partial f}{\partial \mathrm{F}_{02}}$$
$$+ 2\mathrm{F}\frac{\partial f}{\partial \mathrm{G}_{01}} + 2\mathrm{F}_{10}\frac{\partial f}{\partial \mathrm{G}_{11}} + (\mathrm{G}_{10} + 4\mathrm{F}_{01})\frac{\partial f}{\partial \mathrm{G}_{02}}$$
$$+ \phi_{10}\frac{\partial f}{\partial \phi_{02}} + \phi_{20}\frac{\partial f}{\partial \phi_{12}} + 3\phi_{11}\frac{\partial f}{\partial \phi_{03}} + \psi_{10}\frac{\partial f}{\partial \psi_{02}} + \psi_{20}\frac{\partial f}{\partial \psi_{12}} + 3\psi_{11}\frac{\partial f}{\partial \psi_{03}} = 0 \quad . \quad . \quad (\mathrm{II}_3),$$

$$2\mathrm{F}\frac{\partial f}{\partial \mathrm{E}_{10}} + (\mathrm{E}_{01} + 4\mathrm{F}_{10})\frac{\partial f}{\partial \mathrm{E}_{20}} + 2\mathrm{F}_{01}\frac{\partial f}{\partial \mathrm{E}_{11}}$$
$$+ \mathrm{G}\frac{\partial f}{\partial \mathrm{F}_{10}} + (2\mathrm{G}_{10} + \mathrm{F}_{01})\frac{\partial f}{\partial \mathrm{F}_{20}} + \mathrm{G}_{01}\frac{\partial f}{\partial \mathrm{F}_{11}}$$
$$+ \mathrm{G}_{01}\frac{\partial f}{\partial \mathrm{G}_{20}}$$
$$+ \phi_{01}\frac{\partial f}{\partial \phi_{20}} + 3\phi_{11}\frac{\partial f}{\partial \phi_{30}} + \phi_{02}\frac{\partial f}{\partial \phi_{21}} + \psi_{01}\frac{\partial f}{\partial \psi_{20}} + 3\psi_{11}\frac{\partial f}{\partial \psi_{30}} + \psi_{02}\frac{\partial f}{\partial \psi_{21}} = 0 \quad . \quad . \quad (\mathrm{II}_4),$$

$$2\mathrm{F}\frac{\partial f}{\partial \mathrm{E}_{01}} + (\mathrm{E}_{01} + 2\mathrm{F}_{10})\frac{\partial f}{\partial \mathrm{E}_{11}} + 4\mathrm{F}_{01}\frac{\partial f}{\partial \mathrm{E}_{02}}$$
$$+ \mathrm{F}\frac{\partial f}{\partial \mathrm{F}_{10}} + \mathrm{G}\frac{\partial f}{\partial \mathrm{F}_{01}} + 2\mathrm{F}_{10}\frac{\partial f}{\partial \mathrm{F}_{20}} + (2\mathrm{F}_{01} + \mathrm{G}_{10})\frac{\partial f}{\partial \mathrm{F}_{11}} + 2\mathrm{G}_{01}\frac{\partial f}{\partial \mathrm{F}_{02}}$$
$$+ 2\mathrm{G}\frac{\partial f}{\partial \mathrm{G}_{10}} + 4\mathrm{G}_{10}\frac{\partial f}{\partial \mathrm{G}_{20}} + 3\mathrm{G}_{01}\frac{\partial f}{\partial \mathrm{G}_{11}}$$
$$+ \phi_{01}\frac{\partial f}{\partial \phi_{11}} + 2\phi_{11}\frac{\partial f}{\partial \phi_{21}} + 2\phi_{02}\frac{\partial f}{\partial \phi_{12}} + \psi_{01}\frac{\partial f}{\partial \psi_{11}} + 2\psi_{11}\frac{\partial f}{\partial \psi_{21}} + 2\psi_{02}\frac{\partial f}{\partial \psi_{12}} = 0 \quad . \quad (\mathrm{II}_5),$$

$$\mathrm{E}_{01}\frac{\partial f}{\partial \mathrm{E}_{02}}$$
$$+ \mathrm{F}\frac{\partial f}{\partial \mathrm{F}_{01}} + \mathrm{F}_{10}\frac{\partial f}{\partial \mathrm{F}_{11}} + 3\mathrm{F}_{01}\frac{\partial f}{\partial \mathrm{F}_{02}}$$
$$+ 2\mathrm{G}\frac{\partial f}{\partial \mathrm{G}_{01}} + 2\mathrm{G}_{10}\frac{\partial f}{\partial \mathrm{G}_{11}} + 5\mathrm{G}_{01}\frac{\partial f}{\partial \mathrm{G}_{02}}$$
$$+ \phi_{01}\frac{\partial f}{\partial \phi_{02}} + \phi_{11}\frac{\partial f}{\partial \phi_{12}} + 3\phi_{02}\frac{\partial f}{\partial \phi_{03}} + \psi_{01}\frac{\partial f}{\partial \psi_{02}} + \psi_{11}\frac{\partial f}{\partial \psi_{12}} + 3\psi_{02}\frac{\partial f}{\partial \psi_{03}} = 0 \quad . \quad (\mathrm{II}_6),$$

which come from the coefficients of ξ_{20}, ξ_{11}, ξ_{02}, η_{20}, η_{11}, η_{02} respectively; and

$$2E\frac{\partial f}{\partial E_{20}} + F\frac{\partial f}{\partial F_{20}} + \phi_{10}\frac{\partial f}{\partial \phi_{30}} + \psi_{10}\frac{\partial f}{\partial \psi_{30}} = 0 \quad \text{(III}_1\text{)},$$

$$2E\frac{\partial f}{\partial E_{11}} + E\frac{\partial f}{\partial F_{20}} + F\frac{\partial f}{\partial F_{11}} + 2F\frac{\partial f}{\partial G_{20}} + \phi_{10}\frac{\partial f}{\partial \phi_{21}} + \psi_{10}\frac{\partial f}{\partial \psi_{21}} = 0 \quad \text{(III}_2\text{)},$$

$$2E\frac{\partial f}{\partial E_{02}} + E\frac{\partial f}{\partial F_{11}} + F\frac{\partial f}{\partial F_{02}} + 2F\frac{\partial f}{\partial G_{11}} + \phi_{10}\frac{\partial f}{\partial \phi_{12}} + \psi_{10}\frac{\partial f}{\partial \psi_{12}} = 0 \quad \text{(III}_3\text{)},$$

$$E\frac{\partial f}{\partial F_{02}} + 2F\frac{\partial f}{\partial G_{02}} + \phi_{10}\frac{\partial f}{\partial \phi_{03}} + \psi_{10}\frac{\partial f}{\partial \psi_{03}} = 0 \quad \text{(III}_4\text{)},$$

$$2F\frac{\partial f}{\partial E_{20}} + G\frac{\partial f}{\partial F_{20}} + \phi_{10}\frac{\partial f}{\partial \phi_{30}} + \psi_{01}\frac{\partial f}{\partial \psi_{30}} = 0 \quad \text{(III}_5\text{)},$$

$$2F\frac{\partial f}{\partial E_{11}} + F\frac{\partial f}{\partial F_{20}} + G\frac{\partial f}{\partial F_{11}} + 2G\frac{\partial f}{\partial G_{20}} + \phi_{01}\frac{\partial f}{\partial \phi_{21}} + \psi_{01}\frac{\partial f}{\partial \psi_{21}} = 0 \quad \text{(III}_6\text{)},$$

$$2F\frac{\partial f}{\partial E_{02}} + F\frac{\partial f}{\partial F_{11}} + G\frac{\partial f}{\partial F_{02}} + 2G\frac{\partial f}{\partial G_{11}} + \phi_{01}\frac{\partial f}{\partial \phi_{12}} + \psi_{10}\frac{\partial f}{\partial \psi_{12}} = 0 \quad \text{(III}_7\text{)},$$

$$F\frac{\partial f}{\partial F_{02}} + 2G\frac{\partial f}{\partial G_{02}} + \phi_{01}\frac{\partial f}{\partial \phi_{03}} + \psi_{01}\frac{\partial f}{\partial \psi_{03}} = 0 \quad \text{(III}_8\text{)},$$

which come from the coefficients of ξ_{30}, ξ_{21}, ξ_{12}, ξ_{03}, η_{30}, η_{21}, η_{12}, η_{03} respectively.

15. Consider the set of equations (III$_1$) to (III$_8$); all the Poisson-Jacobi conditions of coexistence are satisfied so that, in so far as the third derivatives of ϕ and ψ and the second derivatives of E, F, G are concerned, the set may be regarded as a complete Jacobian system. The total number of variables occurring in the derivatives of f is

4, for the derivatives of ϕ of the third order,

+ 4, ... ψ . . . . . .

+ 9, ... E, F, G of the second order,

= 17 in all; hence as the total number of equations is 8, there will be *nine* algebraically independent solutions involving these 17 quantities. When we integrate the set of equations in the usual manner, we find a set of nine solutions, apparently in their simplest form when given by

$$\left.\begin{aligned}
u_1 &= 2V^2\phi_{30} - (2F_{20} - E_{11})r - E_{20}s \\
u_2 &= 2V^2\phi_{21} - G_{20}r - E_{11}s \\
u_3 &= 2V^2\phi_{12} - G_{11}r - E_{02}s \\
u_4 &= 2V^2\phi_{03} - G_{02}r - (2F_{02} - G_{11})s \\
v_1 &= 2V^2\psi_{30} - (2F_{20} - E_{11})\rho - E_{20}\sigma \\
v_2 &= 2V^2\psi_{21} - G_{20}\rho - E_{11}\sigma \\
v_3 &= 2V^2\psi_{12} - G_{11}\rho - E_{02}\sigma \\
v_4 &= 2V^2\psi_{03} - G_{02}\rho - (2F_{02} - G_{11})\sigma \\
\theta &= E_{02} - 2F_{11} + G_{20}
\end{aligned}\right\},$$

where $V^2 = EG - F^2$, and

$$\left.\begin{aligned} E\phi_{01} - F\phi_{10} &= r \\ G\phi_{10} - F\phi_{01} &= s \end{aligned}\right\}, \qquad \left.\begin{aligned} E\psi_{01} - F\psi_{10} &= \rho \\ G\psi_{10} - F\psi_{01} &= \sigma \end{aligned}\right\}.$$

Any functional combination of these nine quantities will satisfy the set of eight equations which have been considered, as will also any functional combination of the derivatives of ϕ and ψ of orders lower than 3, of the derivatives of E, F, G of orders lower than 2, and of L, M, N, P, Q, R, S, α, β, γ, δ, ϵ. We therefore have to find the functional combinations which will satisfy the remaining equations.

16. For this purpose, we make u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, θ; E, E_{10}, E_{01}; F, F_{10}, F_{01}; G, G_{10}, G_{01}; ϕ_{10}, ϕ_{01}, ϕ_{20}, ϕ_{11}, ϕ_{02}; ψ_{10}, ψ_{01}, ψ_{20}, ψ_{11}, ψ_{02}, the variables; and we transform the set of equations $(II_1)\ldots(II_6)$, so that the derivatives of f are taken with regard to these variables. Denoting f with the new variable by $\bar{f}$ for a moment, we have

$$\frac{\partial f}{\partial \xi} = \frac{\partial \bar{f}}{\partial \xi} + \Sigma \frac{\partial \bar{f}}{\partial u_n}\frac{\partial u_n}{\partial \xi} + \Sigma \frac{\partial \bar{f}}{\partial v_n}\frac{\partial v_n}{\partial \xi} + \frac{\partial \bar{f}}{\partial \theta}\frac{\partial \theta}{\partial \xi},$$

for all the quantities ξ in the original equations; the magnitude $\partial\bar{f}/\partial\xi$ is zero if ξ be not one of the new variables.

The result of the transformation is to replace the set of equations $(II_1)\ldots(II_6)$ by the set

$$\begin{aligned}
&2E\frac{\partial f}{\partial E_{10}} + F\frac{\partial f}{\partial F_{10}} + \phi_{10}\frac{\partial f}{\partial \phi_{20}} + \psi_{10}\frac{\partial f}{\partial \psi_{20}} + (G_{10} - 2F_{10})\frac{\partial f}{\partial \theta} \\
&+ \frac{\partial f}{\partial u_1}[6V^2\phi_{20} + (2E_{01} - 6F_{10})r - 5E_{10}s] + \frac{\partial f}{\partial u_2}[2V^2\phi_{11} - G_{10}r - 2E_{01}s] \\
&+ \frac{\partial f}{\partial v_1}[6V^2\psi_{20} + (2E_{01} - 6F_{10})\rho - 5E_{10}\sigma] + \frac{\partial f}{\partial v_2}[2V^2\psi_{11} - G_{10}\rho - 2E_{01}\sigma] = 0 \quad . \quad (II_1)',
\end{aligned}$$

$$2F\frac{\partial f}{\partial E_{10}} + G\frac{\partial f}{\partial F_{10}} + \phi_{01}\frac{\partial f}{\partial \phi_{20}} + \psi_{01}\frac{\partial f}{\partial \psi_{20}} - G_{01}\frac{\partial f}{\partial \theta}$$
$$+ \frac{\partial f}{\partial u_1}[6V^2\phi_{11} - 4G_{10}r - (E_{01} + 4F_{10})s] + \frac{\partial f}{\partial u_2}[2V^2\phi_{02} - G_{01}r - 2F_{01}s]$$
$$+ \frac{\partial f}{\partial v_1}[6V^2\psi_{11} - 4G_{10}\rho - (E_{01} + 4F_{10})\sigma] + \frac{\partial f}{\partial v_2}[2V^2\psi_{02} - G_{01}\rho - 2F_{01}\sigma] = 0 \quad . \quad (\mathrm{II}_4)',$$

$$2E\frac{\partial f}{\partial E_{01}} + E\frac{\partial f}{\partial F_{10}} + F\frac{\partial f}{\partial F_{01}} + 2F\frac{\partial f}{\partial G_{10}} + \phi_{10}\frac{\partial f}{\partial \phi_{11}} + \psi_{10}\frac{\partial f}{\partial \psi_{11}} + 2E_{01}\frac{\partial f}{\partial \theta}$$
$$- E_{10}r\frac{\partial f}{\partial u_1} + \frac{\partial f}{\partial u_2}[4V^2\phi_{20} - 4F_{10}r - 3E_{10}s] + \frac{\partial f}{\partial u_3}[4V^2\phi_{11} - (G_{10} + 2F_{01})r - 4E_{01}s]$$
$$+ (G_{10} - 2F_{01})s\frac{\partial f}{\partial u_4}$$
$$- E_{10}\rho\frac{\partial f}{\partial v_1} + \frac{\partial f}{\partial v_2}[4V^2\psi_{20} - 4F_{10}\rho - 3E_{10}\sigma] + \frac{\partial f}{\partial v_3}[4V^2\psi_{11} - (G_{10} + 2F_{01})\rho - 4E_{01}\sigma]$$
$$+ (G_{10} - 2F_{01})\sigma\frac{\partial f}{\partial v_4} = 0 \quad . \quad (\mathrm{II}_2)',$$

$$2F\frac{\partial f}{\partial E_{01}} + F\frac{\partial f}{\partial F_{10}} + G\frac{\partial f}{\partial F_{01}} + 2G\frac{\partial f}{\partial G_{10}} + \phi_{01}\frac{\partial f}{\partial \phi_{11}} + \psi_{01}\frac{\partial f}{\partial \psi_{11}} + 2G_{10}\frac{\partial f}{\partial \theta}$$
$$+ (E_{01} - 2F_{10})r\frac{\partial f}{\partial u_1} + \frac{\partial f}{\partial u_2}[4V^2\phi_{11} - 4G_{10}r - (E_{01} + 2F_{10})s]$$
$$+ \frac{\partial f}{\partial u_3}[4V^2\phi_{02} - 3G_{01}r - 4F_{01}s] - G_{01}s\frac{\partial f}{\partial u_4}$$
$$+ (E_{01} - 2F_{10})\rho\frac{\partial f}{\partial v_1} + \frac{\partial f}{\partial v_2}[4V^2\psi_{11} - 4G_{10}\rho - (E_{01} + 2F_{10})\sigma]$$
$$+ \frac{\partial f}{\partial v_3}[4V^2\psi_{02} - 3G_{01}\rho - 4F_{01}\sigma] - G_{01}\sigma\frac{\partial f}{\partial v_4} = 0 \quad . \quad (\mathrm{II}_5)',$$

$$E\frac{\partial f}{\partial F_{01}} + 2F\frac{\partial f}{\partial G_{01}} + \phi_{10}\frac{\partial f}{\partial \phi_{02}} + \psi_{10}\frac{\partial f}{\partial \psi_{02}} - E_{10}\frac{\partial f}{\partial \theta}$$
$$+ \frac{\partial f}{\partial u_3}[2V^2\phi_{20} - 2F_{10}r - E_{10}s] + \frac{\partial f}{\partial u_4}[6V^2\phi_{11} - (G_{10} + 4F_{01})r - 4E_{01}s]$$
$$+ \frac{\partial f}{\partial v_3}[2V^2\psi_{20} - 2F_{10}\rho - E_{10}\sigma] + \frac{\partial f}{\partial v_4}[6V^2\phi_{11} - (G_{10} + 4F_{01})\rho - 4E_{01}\sigma] = 0 \quad . \quad (\mathrm{II}_3)',$$

$$F\frac{\partial f}{\partial F_{01}} + 2G\frac{\partial f}{\partial G_{01}} + \phi_{01}\frac{\partial f}{\partial \phi_{02}} + \psi_{01}\frac{\partial f}{\partial \psi_{02}} + (E_{01} - 2F_{10})\frac{\partial f}{\partial \theta}$$
$$+ \frac{\partial f}{\partial u_3}[2V^2\phi_{11} - 2G_{10}r - E_{01}s] + \frac{\partial f}{\partial u_4}[6V^2\phi_{02} - 5G_{01}r + (2G_{10} - 6F_{01})s]$$
$$+ \frac{\partial f}{\partial v_3}[2V^2\psi_{11} - 2G_{10}\rho - E_{01}\sigma] + \frac{\partial f}{\partial v_4}[6V^2\psi_{02} - 5G_{01}\rho + (2G_{10} - 6F_{01})\sigma] = 0 \quad . \quad (\mathrm{II}_6)'.$$

17. A special case of these six equations is discussed* by Professor Žorawski in his memoir already quoted, viz., that in which there occurs a single function ϕ with its derivatives up to the second order inclusive, and there are no derivatives of E, F, G of order higher than the first; and he obtains three independent solutions. These are

$$\left.\begin{aligned} a &= 2V^2\phi_{20} + (E_{01} - 2F_{10})\, r - E_{10}s \\ b &= 2V^2\phi_{11} - G_{10}r - E_{01}s \\ c &= 2V^2\phi_{02} - G_{01}r + (G_{10} - 2F_{01})\, s \end{aligned}\right\}$$

Manifestly, a, b, c are independent solutions of the equations in the present case; also, other three independent solutions are given by

$$\left.\begin{aligned} a' &= 2V^2\psi_{20} + (E_{01} - 2F_{10})\, \rho - E_{10}\sigma \\ b' &= 2V^2\psi_{11} - G_{10}\rho - E_{01}\sigma \\ c' &= 2V^2\psi_{02} - G_{01}\rho + (G_{10} - 2F_{01})\, \sigma \end{aligned}\right\}.$$

All these six solutions are independent of θ; u_1, u_2, u_3, u_4; v_1, v_2, v_3, v_4.

The Jacobi-Poisson conditions of coexistence of the six equations are satisfied either identically or in virtue of the eight equations (III_1) to (III_8), which are definitely satisfied; so that, taking account of the variables that occur in the derivatives of f, the set of six equations is a complete system. The number of these variables is

6, from the first derivatives of E, F, G,
+ 6, . . . second . . . . ϕ, ψ,
+ 1, being θ,
+ 8, being u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4,

= 21 in all; hence the total number of algebraically independent solutions of the complete system of six equations is 15. Of these, we already possess six in a, b, c, a', b', c', so that other nine are required.

The form of the equations suggests that there will be four solutions of the type

$$u_n + aP_n + bQ_n + cR_n + S_n,$$

four of the type

$$v_n + a'P'_n + b'Q'_n + c'R'_n + S'_n,$$

and one of the type

$$\theta + T_n;$$

which, when obtained, will be the necessary nine.

18. One mode of obtaining these solutions is as follows:—We use the values of a, b, c, a', b', c' to eliminate from f the second derivatives of ϕ and of ψ; the effect is

* *Loc. cit.*, § 26.

to modify the form of the equations $(\mathrm{II}_1)' \ldots (\mathrm{II}_6)'$ by removing from them all the terms that involve those derivatives. The substituted derivatives with respect to a, b, c, a', b', c' do not occur—a result only to be expected, because these quantities are simultaneous solutions of all the six equations. Consequently, in any differential operations, the quantities a, b, c, a', b', c' behave like constants.

In order that

$$f = u_1 + a\mathrm{P}_1 + b\mathrm{Q}_1 + c\mathrm{R}_1 + \mathrm{S}_1,$$

where $\mathrm{P}_1, \mathrm{Q}_1, \mathrm{R}_1, \mathrm{S}_1$ are functions of ϕ_{10}, ϕ_{01}, and of the first derivatives of E, F, G, but are independent of the quantities u and v, θ, a, b, c, a', b', c', may satisfy $(\mathrm{II}_1)'$, we must have

$$\left(2\mathrm{E}\frac{\partial}{\partial \mathrm{E}_{10}} + \mathrm{F}\frac{\partial}{\partial \mathrm{F}_{10}}\right)\mathrm{P}_1 = -3,$$

$$\left(2\mathrm{E}\frac{\partial}{\partial \mathrm{E}_{10}} + \mathrm{F}\frac{\partial}{\partial \mathrm{F}_{10}}\right)\mathrm{Q}_1 = 0,$$

$$\left(2\mathrm{E}\frac{\partial}{\partial \mathrm{E}_{10}} + \mathrm{F}\frac{\partial}{\partial \mathrm{F}_{10}}\right)\mathrm{R}_1 = 0,$$

$$\left(2\mathrm{E}\frac{\partial}{\partial \mathrm{E}_{10}} + \mathrm{F}\frac{\partial}{\partial \mathrm{F}_{10}}\right)\mathrm{S}_1 = \mathrm{E}_{01}r + 2\mathrm{E}_{10}s.$$

In order that the same quantity may satisfy $(\mathrm{II}_4)'$, we must have

$$\left(2\mathrm{F}\frac{\partial}{\partial \mathrm{E}_{10}} + \mathrm{G}\frac{\partial}{\partial \mathrm{F}_{10}}\right)\mathrm{P}_1 = 0,$$

$$\left(2\mathrm{F}\frac{\partial}{\partial \mathrm{E}_{10}} + \mathrm{G}\frac{\partial}{\partial \mathrm{F}_{10}}\right)\mathrm{Q}_1 = -3,$$

$$\left(2\mathrm{F}\frac{\partial}{\partial \mathrm{E}_{10}} + \mathrm{G}\frac{\partial}{\partial \mathrm{F}_{10}}\right)\mathrm{R}_1 = 0,$$

$$\left(2\mathrm{F}\frac{\partial}{\partial \mathrm{E}_{10}} + \mathrm{G}\frac{\partial}{\partial \mathrm{F}_{10}}\right)\mathrm{S}_1 = \mathrm{G}_{10}r - 2(\mathrm{E}_{01} - 2\mathrm{F}_{10})s.$$

Similarly, the equation $(\mathrm{II}_3)'$ requires

$$\left(\mathrm{E}\frac{\partial}{\partial \mathrm{F}_{01}} + 2\mathrm{F}\frac{\partial}{\partial \mathrm{G}_{01}}\right)\Theta = 0,$$

and the equation $(\mathrm{II}_6)'$ requires

$$\left(\mathrm{F}\frac{\partial}{\partial \mathrm{F}_{01}} + 2\mathrm{G}\frac{\partial}{\partial \mathrm{G}_{01}}\right)\Theta = 0,$$

for $\Theta = \mathrm{P}_1, \mathrm{Q}_1, \mathrm{R}_1, \mathrm{S}_1$. The equation $(\mathrm{II}_2)'$ requires

$$\left(2\mathrm{E}\frac{\partial}{\partial \mathrm{E}_{01}} + \mathrm{E}\frac{\partial}{\partial \mathrm{F}_{10}} + \mathrm{F}\frac{\partial}{\partial \mathrm{F}_{01}} + 2\mathrm{F}\frac{\partial}{\partial \mathrm{G}_{10}}\right)\Phi = 0$$

for $\Phi = \mathrm{P}_1, \mathrm{Q}_1, \mathrm{R}_1$, and

$$\left(2E\frac{\partial}{\partial E_{01}} + E\frac{\partial}{\partial F_{10}} + F\frac{\partial}{\partial F_{01}} + 2F\frac{\partial}{\partial G_{10}}\right) S_1 = E_{10} r ;$$

and the equation $(II_5)'$ requires

$$\left(2F\frac{\partial}{\partial E_{01}} + F\frac{\partial}{\partial F_{10}} + G\frac{\partial}{\partial F_{01}} + 2G\frac{\partial}{\partial G_{10}}\right) \Phi = 0 ,$$

for $\Phi = P_1, Q_1, R_1$, and

$$\left(2F\frac{\partial}{\partial E_{01}} + F\frac{\partial}{\partial F_{10}} + G\frac{\partial}{\partial F_{01}} + 2G\frac{\partial}{\partial G_{10}}\right) S_1 = -(E_{01} - 2F_{10})\, r .$$

We thus have 24 equations giving the derivatives of the four quantities P_1, Q_1, R_1, S_1 with respect to $E_{10}, E_{01}, F_{10}, F_{01}, G_{10}, G_{01}$. Each of the four quantities is then given by effecting the quadrature

$$\Theta = \int \left(\frac{\partial \Theta}{\partial E_{10}} dE_{10} + . \quad . \quad . + \frac{\partial \Theta}{\partial G_{01}} dG_{01}\right).$$

The results are

$$2V^2 P_1 = 3(-E_{10}G - E_{01}F + 2F_{10}F),$$
$$2V^2 Q_1 = 3(E_{10}F + E_{01}E - 2F_{10}E),$$
$$2V^2 R_1 = 0,$$
$$\begin{aligned} 2V^2 S_1 = {} & \{EG_{10}(2F_{10} - E_{01}) + F(E_{01}^2 - 2E_{01}F_{10} - E_{10}G_{10}) + GE_{10}E_{01}\}\, r \\ & + \{E(E_{01}^2 - 4E_{01}F_{10} + 4F_{10}^2) + 2FE_{10}(E_{01} - 2F_{10}) + GE_{10}^2\}\, s. \end{aligned}$$

The solution in question remains a solution when it is multiplied by $2V^2$; denoting this product by κ', we have

$$\kappa' = 2V^2 u_1 + ap_1 + bq_1 + cr_1 + re_1 + sf_1.$$

Similarly we obtain

$$\left.\begin{aligned} \lambda' &= 2V^2 u_2 + ap_2 + bq_2 + cr_2 + re_2 + sf_2 \\ \mu' &= 2V^2 u_3 + ap_3 + bq_3 + cr_3 + re_3 + sf_3 \\ \nu' &= 2V^2 u_4 + ap_4 + bq_4 + cr_4 + re_4 + sf_4 \end{aligned}\right\},$$

where

$$\left.\begin{aligned} p_1 &= 3\{-F(E_{01} - 2F_{10}) - GE_{10}\} \\ q_1 &= 3\{E(E_{01} - 2F_{10}) + FE_{10}\} \\ r_1 &= 0 \\ e_1 &= -EG_{10}(E_{01} - 2F_{10}) + F(E_{01}^2 - 2E_{01}F_{10} - E_{10}G_{10}) + GE_{10}E_{01} \\ f_1 &= E(E_{01} - 2F_{10})^2 + 2FE_{10}(E_{01} - 2F_{10}) + GE_{10}^2 \end{aligned}\right\},$$

$$\left.\begin{aligned} p_2 &= 2FG_{10} - 2GE_{01} \\ q_2 &= -2EG_{10} + F(E_{01} + 2F_{10}) - GE_{10} \\ r_2 &= E(E_{01} - 2F_{10}) + FE_{10} \\ e_2 &= EG_{10}^2 - 2FE_{01}G_{10} + GE_{01}^2 \\ f_2 &= e_1, \text{ above} \end{aligned}\right\},$$

$$\left.\begin{aligned} p_3 &= FG_{01} - G(2F_{01} - G_{10}) \\ q_3 &= -EG_{01} + F(2F_{01} + G_{10}) - 2GE_{01} \\ r_3 &= -2EG_{10} + 2FE_{01} \\ e_3 &= EG_{10}G_{01} + F(-E_{01}G_{01} - 2F_{01}G_{10} + G_{10}^2) + GE_{01}(2F_{01} - G_{10}) \\ f_3 &= e_2, \text{ above} \end{aligned}\right\},$$

$$\left.\begin{aligned} p_4 &= 0 \\ q_4 &= 3\{FG_{01} - G(2F_{01} - G_{10})\} \\ r_4 &= 3\{-EG_{01} + F(2F_{01} - G_{10})\} \\ e_4 &= EG_{01}^2 - 2FG_{01}(2F_{01} - G_{10}) + G(2F_{01} - G_{10})^2 \\ f_4 &= e_3, \text{ above} \end{aligned}\right\}.$$

Other four solutions are given by

$$\left.\begin{aligned} \kappa'' &= 2V^2v_1 + a'p_1 + b'q_1 + c'r_1 + \rho e_1 + \sigma f_1 \\ \lambda'' &= 2V^2v_2 + a'p_2 + b'q_2 + c'r_2 + \rho e_2 + \sigma f_2 \\ \mu'' &= 2V^2v_3 + a'p_3 + b'q_3 + c'r_3 + \rho e_3 + \sigma f_3 \\ \nu'' &= 2V^2v_4 + a'p_4 + b'q_4 + c'r_4 + \rho e_4 + \sigma f_4 \end{aligned}\right\};$$

and there is a last solution given by

$$\begin{aligned} \nabla = E\,&\{(E_{01} - 2F_{10})G_{01} + G_{10}^2\} \\ &+ F\{E_{10}G_{01} - E_{01}(2F_{01} + G_{10}) + 2F_{10}(2F_{01} - G_{10})\} \\ &+ G\{E_{01}^2 - E_{10}(2F_{01} - G_{10})\} \\ &- 2V^2(E_{02} - 2F_{11} + G_{20}). \end{aligned}$$

Consequently, it follows that every simultaneous solution of the fourteen equations made up of the eight $(III_1) \ldots (III_8)$ and of the six $(II_1) \ldots (II_6)$, is a functional combination of the fifteen quantities

$$a,\ b,\ c,\ a',\ b',\ c',$$

$$\kappa',\ \lambda',\ \mu',\ \nu',\ \kappa'',\ \lambda'',\ \mu'',\ \nu'',$$

$$\nabla,$$

and of the quantities derivatives with regard to which have not occurred in those fourteen equations, viz., E, F, G, L, M, N, P, Q, R, S, α, β, γ, δ, ϵ, ϕ_{10}, ϕ_{01}, ψ_{10}, ψ_{01}, making 34 arguments in all What now is required is the algebraically independent aggregate of the functional combinations of these 34 arguments satisfying the remaining four differential equations $(I_1) \ldots (I_4)$.

19. As regards these equations, we replace (I_1) and (I_2) by two equations composed of their sum and their difference. The former is

$$
\begin{aligned}
2\mu f = {} & 2\left(E\frac{\partial f}{\partial E} + F\frac{\partial f}{\partial F} + G\frac{\partial f}{\partial G}\right) + 2\left(L\frac{\partial f}{\partial L} + M\frac{\partial f}{\partial M} + N\frac{\partial f}{\partial N}\right) \\
& + 3\left(P\frac{\partial f}{\partial P} + Q\frac{\partial f}{\partial Q} + R\frac{\partial f}{\partial R} + S\frac{\partial f}{\partial S}\right) \\
& + 4\left(\alpha\frac{\partial f}{\partial \alpha} + \beta\frac{\partial f}{\partial \beta} + \gamma\frac{\partial f}{\partial \gamma} + \delta\frac{\partial f}{\partial \delta} + \epsilon\frac{\partial f}{\partial \epsilon}\right) \\
& + 3\left(E_{10}\frac{\partial f}{\partial E_{10}} + E_{01}\frac{\partial f}{\partial E_{01}} + F_{10}\frac{\partial f}{\partial F_{10}} + F_{01}\frac{\partial f}{\partial F_{01}} + G_{10}\frac{\partial f}{\partial G_{10}} + G_{01}\frac{\partial f}{\partial G_{01}}\right) \\
& + 4\left(E_{20}\frac{\partial f}{\partial E_{20}} + E_{11}\frac{\partial f}{\partial E_{11}} + E_{02}\frac{\partial f}{\partial E_{02}} + F_{20}\frac{\partial f}{\partial F_{20}} + F_{11}\frac{\partial f}{\partial F_{11}} + F_{02}\frac{\partial f}{\partial F_{02}}\right. \\
& \qquad \left. + G_{20}\frac{\partial f}{\partial G_{20}} + G_{11}\frac{\partial f}{\partial G_{11}} + G_{02}\frac{\partial f}{\partial G_{02}}\right) \\
& + \phi_{10}\frac{\partial f}{\partial \phi_{10}} + \phi_{01}\frac{\partial f}{\partial \phi_{01}} + \psi_{10}\frac{\partial f}{\partial \psi_{10}} + \psi_{01}\frac{\partial f}{\partial \psi_{01}} \\
& + 2\left(\phi_{20}\frac{\partial f}{\partial \phi_{20}} + \phi_{11}\frac{\partial f}{\partial \phi_{11}} + \phi_{02}\frac{\partial f}{\partial \phi_{02}} + \psi_{20}\frac{\partial f}{\partial \psi_{20}} + \psi_{11}\frac{\partial f}{\partial \psi_{11}} + \psi_{02}\frac{\partial f}{\partial \psi_{02}}\right) \\
& + 3\left(\phi_{30}\frac{\partial f}{\partial \phi_{30}} + \phi_{21}\frac{\partial f}{\partial \phi_{21}} + \phi_{12}\frac{\partial f}{\partial \phi_{12}} + \phi_{03}\frac{\partial f}{\partial \phi_{03}}\right. \\
& \qquad \left. + \psi_{30}\frac{\partial f}{\partial \psi_{30}} + \psi_{21}\frac{\partial f}{\partial \psi_{21}} + \psi_{12}\frac{\partial f}{\partial \psi_{12}} + \psi_{03}\frac{\partial f}{\partial \psi_{03}}\right) \quad . \quad . \quad (I_1).
\end{aligned}
$$

The latter is

$$
\begin{aligned}
0 = {} & 2\left(E\frac{\partial f}{\partial E} - G\frac{\partial f}{\partial G} + L\frac{\partial f}{\partial L} - N\frac{\partial f}{\partial N}\right) + 3P\frac{\partial f}{\partial P} + Q\frac{\partial f}{\partial Q} - R\frac{\partial f}{\partial R} - 3S\frac{\partial f}{\partial S} \\
& + 4\alpha\frac{\partial f}{\partial \alpha} + 2\beta\frac{\partial f}{\partial \beta} - 2\delta\frac{\partial f}{\partial \delta} - 4\epsilon\frac{\partial f}{\partial \epsilon} \\
& + 3E_{10}\frac{\partial f}{\partial E_{10}} + E_{01}\frac{\partial f}{\partial E_{01}} + F_{10}\frac{\partial f}{\partial F_{10}} - F_{01}\frac{\partial f}{\partial F_{01}} - G_{10}\frac{\partial f}{\partial G_{10}} - 3G_{01}\frac{\partial f}{\partial G_{01}} \\
& + 4E_{20}\frac{\partial f}{\partial E_{20}} + 2E_{11}\frac{\partial f}{\partial E_{11}} + 2F_{20}\frac{\partial f}{\partial F_{20}} - 2F_{02}\frac{\partial f}{\partial F_{02}} - 2G_{11}\frac{\partial f}{\partial G_{11}} - 4G_{02}\frac{\partial f}{\partial G_{02}} \\
& + \phi_{10}\frac{\partial f}{\partial \phi_{10}} - \phi_{01}\frac{\partial f}{\partial \phi_{01}} + \psi_{10}\frac{\partial f}{\partial \psi_{10}} - \psi_{01}\frac{\partial f}{\partial \psi_{01}} \\
& + 2\left(\phi_{20}\frac{\partial f}{\partial \phi_{20}} - \phi_{02}\frac{\partial f}{\partial \phi_{02}} + \psi_{20}\frac{\partial f}{\partial \psi_{20}} - \psi_{02}\frac{\partial f}{\partial \psi_{02}}\right) \\
& + 3\phi_{30}\frac{\partial f}{\partial \phi_{30}} + \phi_{21}\frac{\partial f}{\partial \phi_{21}} - \phi_{12}\frac{\partial f}{\partial \phi_{12}} - 3\phi_{03}\frac{\partial f}{\partial \phi_{03}} \\
& + 3\psi_{30}\frac{\partial f}{\partial \psi_{30}} + \psi_{21}\frac{\partial f}{\partial \psi_{21}} - \psi_{12}\frac{\partial f}{\partial \psi_{12}} - 3\psi_{03}\frac{\partial f}{\partial \psi_{03}} \quad . \quad . \quad . \quad . \quad (I_2).
\end{aligned}
$$

Of these four equations $(\mathrm{I}_1)'$, $(\mathrm{I}_2)'$, (I_3), (I_4), the first will be found to be satisfied for the various forms of f that satisfy the other three, by the appropriate determination of the constant μ to be associated with each such form. Also, $(\mathrm{I}_2)'$ is the condition to be satisfied in order that (I_3) and (I_4) may possess common solutions. To obtain these common solutions, we proceed as follows.

Let the equations (I_3) and (I_4) be written

$$\nabla_1 f = 0, \quad \nabla_2 f = 0.$$

Then by actual substitution we obtain the results

$$\left.\begin{array}{ll} \nabla_1 a = 0, & \nabla_2 a = 2b \\ \nabla_1 b = a, & \nabla_2 b = c \\ \nabla_1 c = 2b, & \nabla_2 c = 0 \end{array}\right\};$$

$$\left.\begin{array}{ll} \nabla_1 a' = 0, & \nabla_2 a' = 2b' \\ \nabla_1 b' = a', & \nabla_2 b' = c' \\ \nabla_1 c' = 2b', & \nabla_2 c' = 0 \end{array}\right\};$$

$$\nabla_1 \nabla = 0, \qquad \nabla_2 \nabla = 0.$$

Also

$$\begin{array}{ll} \nabla_1 \kappa' = 0, & \nabla_2 \kappa' = 3\lambda' - \nabla r, \\ \nabla_1 \lambda' = \kappa', & \nabla_2 \lambda' = 2\mu' - \nabla s, \\ \nabla_1 \mu' = 2\lambda' - \nabla r, & \nabla_2 \mu' = \nu', \\ \nabla_1 \nu' = 3\mu' - \nabla s, & \nabla_2 \nu' = 0, \\ \nabla_1 r = 0, & \nabla_2 r = -s, \\ \nabla_1 s = -r, & \nabla_2 s = 0\,; \end{array}$$

and therefore

$$\begin{array}{ll} \nabla_1 \kappa' = 0, & \nabla_2 \kappa' = 3(\lambda' - \tfrac{1}{3}\nabla r), \\ \nabla_1 (\lambda' - \tfrac{1}{3}\nabla r) = \kappa', & \nabla_2 (\lambda' - \tfrac{1}{3}\nabla r) = 2(\mu' - \tfrac{1}{3}\nabla s), \\ \nabla_1 (\mu' - \tfrac{1}{3}\nabla s) = 2(\lambda' - \tfrac{1}{3}\nabla r), & \nabla_2 (\mu' - \tfrac{1}{3}\nabla s) = \nu', \\ \nabla_1 \nu' = 3(\mu' - \tfrac{1}{3}\nabla s), & \nabla_2 \nu' = 0. \end{array}$$

We write

$$\kappa' = k, \quad \lambda' - \tfrac{1}{3}\nabla r = l, \quad \mu' - \tfrac{1}{3}\nabla s = m, \quad \nu' = n\,;$$

and then these equations give

$$\left.\begin{array}{ll} \nabla_1 k = 0, & \nabla_2 k = 3l \\ \nabla_1 l = k, & \nabla_2 l = 2m \\ \nabla_1 m = 2l, & \nabla_2 m = n \\ \nabla_1 l = 3m, & \nabla_2 n = 0 \end{array}\right\}.$$

Similarly, we write

$$\kappa'' = k', \quad \lambda'' - \tfrac{1}{3}\nabla\rho = l', \quad \mu'' - \tfrac{1}{3}\nabla\sigma = m', \quad \nu'' = n';$$

and we find

$$\left.\begin{aligned} \nabla_1 k' &= 0, & \nabla_2 k' &= 3l' \\ \nabla_1 l' &= k', & \nabla_2 l' &= 2m' \\ \nabla_1 m' &= 2l', & \nabla_2 m' &= n' \\ \nabla_1 n' &= 3m', & \nabla_2 n' &= 0 \end{aligned}\right\}.$$

These quantities $k, l, m, n, k', l', m', n'$ replace $\kappa', \lambda', \mu', \nu', \kappa'', \lambda'', \mu'', \nu''$; moreover, ∇ is a simultaneous solution of the equations. What we require are the functional combinations of the quantities

$$a, b, c, \ a', b', c',$$

$$k, l, m, n, \ k', l', m', n',$$

$$\mathrm{E, F, G, L, M, N, P, Q, R, S}, \ \alpha, \beta, \gamma, \delta, \epsilon, \phi_{10}, \phi_{01}, \psi_{10}, \psi_{01},$$

making 33 arguments in all.

For this purpose, we transform the equations, so that these 33 arguments may become the independent variables. The process would be laborious but not intrinsically difficult, were it not that the effect of the operators ∇_1 and ∇_2 upon the various arguments has already been obtained; and the results are

$$\begin{aligned} & 2\mathrm{F}\frac{\partial f}{\partial \mathrm{G}} + \mathrm{E}\frac{\partial f}{\partial \mathrm{F}} \\ & + 2\mathrm{M}\frac{\partial f}{\partial \mathrm{N}} + \mathrm{L}\frac{\partial f}{\partial \mathrm{M}} \\ & + 3\mathrm{R}\frac{\partial f}{\partial \mathrm{S}} + 2\mathrm{Q}\frac{\partial f}{\partial \mathrm{R}} + \mathrm{P}\frac{\partial f}{\partial \mathrm{Q}} \\ & + 4\delta\frac{\partial f}{\partial \epsilon} + 3\gamma\frac{\partial f}{\partial \delta} + 2\beta\frac{\partial f}{\partial \gamma} + \alpha\frac{\partial f}{\partial \beta} \\ & + 2b\frac{\partial f}{\partial c} + a\frac{\partial f}{\partial b} \\ & + 2b'\frac{\partial f}{\partial c'} + a'\frac{\partial f}{\partial b'} \\ & + 3m\frac{\partial f}{\partial n} + 2l\frac{\partial f}{\partial m} + k\frac{\partial f}{\partial l} \\ & + 3m'\frac{\partial f}{\partial n'} + 2l'\frac{\partial f}{\partial m'} + k'\frac{\partial f}{\partial l'} \\ & + \phi_{10}\frac{\partial f}{\partial \phi_{01}} \\ & + \psi_{10}\frac{\partial f}{\partial \psi_{01}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I}_3)', \end{aligned}$$

$$
\begin{aligned}
&2F\frac{\partial f}{\partial E} + G\frac{\partial f}{\partial F} \\
&+ 2M\frac{\partial f}{\partial L} + N\frac{\partial f}{\partial M} \\
&+ 3Q\frac{\partial f}{\partial P} + 2R\frac{\partial f}{\partial Q} + S\frac{\partial f}{\partial R} \\
&+ 4\beta\frac{\partial f}{\partial \alpha} + 3\gamma\frac{\partial f}{\partial \beta} + 2\delta\frac{\partial f}{\partial \gamma} + \epsilon\frac{\partial f}{\partial \delta} \\
&+ 2b\frac{\partial f}{\partial a} + c\frac{\partial f}{\partial b} \\
&+ 3l\frac{\partial f}{\partial k} + 2m\frac{\partial f}{\partial l} + n\frac{\partial f}{\partial m} \\
&+ 2b'\frac{\partial f}{\partial a'} + c'\frac{\partial f}{\partial b'} \\
&+ 3l'\frac{\partial f}{\partial k'} + 2m'\frac{\partial f}{\partial l'} + n'\frac{\partial f}{\partial m'} \\
&+ \phi_{01}\frac{\partial f}{\partial \phi_{10}} \\
&+ \psi_{01}\frac{\partial f}{\partial \psi_{10}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I}_4)',
\end{aligned}
$$

$$
\begin{aligned}
&\phi_{10}\frac{\partial f}{\partial \phi_{10}} - \phi_{01}\frac{\partial f}{\partial \phi_{01}} + \psi_{10}\frac{\partial f}{\partial \psi_{10}} - \psi_{01}\frac{\partial f}{\partial \psi_{01}} \\
&+ 2\left(E\frac{\partial f}{\partial E} - G\frac{\partial f}{\partial G} + L\frac{\partial f}{\partial L} - N\frac{\partial f}{\partial N} + a\frac{\partial f}{\partial a} - c\frac{\partial f}{\partial c} + a'\frac{\partial f}{\partial a'} - c'\frac{\partial f}{\partial c'}\right) \\
&+ 3P\frac{\partial f}{\partial P} + Q\frac{\partial f}{\partial Q} - R\frac{\partial f}{\partial R} - 3S\frac{\partial f}{\partial S} \\
&+ 3k\frac{\partial f}{\partial k} + l\frac{\partial f}{\partial l} - m\frac{\partial f}{\partial m} - 3n\frac{\partial f}{\partial n} \\
&+ 3k'\frac{\partial f}{\partial k'} + l'\frac{\partial f}{\partial l'} - m'\frac{\partial f}{\partial m'} - 3n'\frac{\partial f}{\partial n'} \\
&+ 4\alpha\frac{\partial f}{\partial \alpha} + 2\beta\frac{\partial f}{\partial \beta} - 2\delta\frac{\partial f}{\partial \delta} - 4\epsilon\frac{\partial f}{\partial \epsilon} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I}_2)'',
\end{aligned}
$$

$$
\begin{aligned}
2\mu f = {} & \phi_{10}\frac{\partial f}{\partial \phi_{10}} + \phi_{01}\frac{\partial f}{\partial \phi_{01}} + \psi_{10}\frac{\partial f}{\partial \psi_{10}} + \psi_{01}\frac{\partial f}{\partial \psi_{01}} \\
& + 2\left(\mathrm{E}\frac{\partial f}{\partial \mathrm{E}} + \mathrm{F}\frac{\partial f}{\partial \mathrm{F}} + \mathrm{G}\frac{\partial f}{\partial \mathrm{G}} + \mathrm{L}\frac{\partial f}{\partial \mathrm{L}} + \mathrm{M}\frac{\partial f}{\partial \mathrm{M}} + \mathrm{N}\frac{\partial f}{\partial \mathrm{N}}\right) \\
& + 3\left(\mathrm{P}\frac{\partial f}{\partial \mathrm{P}} + \mathrm{Q}\frac{\partial f}{\partial \mathrm{Q}} + \mathrm{R}\frac{\partial f}{\partial \mathrm{R}} + \mathrm{S}\frac{\partial f}{\partial \mathrm{S}}\right) \\
& + 4\left(\alpha\frac{\partial f}{\partial \alpha} + \beta\frac{\partial f}{\partial \beta} + \gamma\frac{\partial f}{\partial \gamma} + \delta\frac{\partial f}{\partial \delta} + \epsilon\frac{\partial f}{\partial \epsilon}\right) \\
& + 6\left(a\frac{\partial f}{\partial a} + b\frac{\partial f}{\partial b} + c\frac{\partial f}{\partial c} + a'\frac{\partial f}{\partial a'} + b'\frac{\partial f}{\partial b'} + c'\frac{\partial f}{\partial c'}\right) \\
& + 8\nabla\frac{\partial f}{\partial \nabla} \\
& + 11\left(k\frac{\partial f}{\partial k} + l\frac{\partial f}{\partial l} + m\frac{\partial f}{\partial m} + n\frac{\partial f}{\partial n}\right) \\
& + 11\left(k'\frac{\partial f}{\partial k'} + l'\frac{\partial f}{\partial l'} + m'\frac{\partial f}{\partial m'} + n'\frac{\partial f}{\partial n'}\right) \quad \ldots \ldots \quad (\mathrm{I}_1)''.
\end{aligned}
$$

Association with Binariants.

20. The expression of these equations at once associates the solution with known results in the theory of the concomitants of a system of simultaneous binary forms. The equations $(\mathrm{I}_3)'$, $(\mathrm{I}_4)'$, $(\mathrm{I}_2)''$ are the differential equations of the invariants of the system of binary forms

$$
\begin{aligned}
& (\phi_{10},\ \phi_{01} \between *)^1,\ \ (\psi_{10},\ \psi_{01} \between *)^1, \\
& (\mathrm{E},\ \mathrm{F},\ \mathrm{G} \between *)^2,\ \ (\mathrm{L},\ \mathrm{M},\ \mathrm{N} \between *)^2,\ \ (a,\ b,\ c \between *)^2,\ \ (a',\ b',\ c' \between *)^2, \\
& (\mathrm{P},\ \mathrm{Q},\ \mathrm{R},\ \mathrm{S} \between *)^3,\ \ (k,\ l,\ m,\ n \between *)^3,\ \ (k',\ l',\ m',\ n' \between *)^3, \\
& (\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon \between *)^4,
\end{aligned}
$$

or, what is the same thing, they are the differential equations of the invariants and covariants of the system of binary forms

$$
\begin{aligned}
w_1 &= (\psi_{10},\ \psi_{01} \between \phi_{01},\ -\phi_{10}), \\
w_2 &= (\mathrm{E},\ \mathrm{F},\ \mathrm{G} \between \phi_{01},\ -\phi_{10})^2, \\
w'_2 &= (\mathrm{L},\ \mathrm{M},\ \mathrm{N} \between \phi_{01},\ -\phi_{10})^2, \\
w''_2 &= (a,\ b,\ c \between \phi_{01},\ -\phi_{10})^2, \\
w'''_2 &= (a',\ b',\ c' \between \phi_{01},\ -\phi_{10})^2, \\
w_3 &= (\mathrm{P},\ \mathrm{Q},\ \mathrm{R},\ \mathrm{S} \between \phi_{01},\ -\phi_{10})^3, \\
w'_3 &= (k,\ l,\ m,\ n \between \phi_{01},\ -\phi_{10})^3, \\
w''_3 &= (k',\ l',\ m',\ n' \between \phi_{01},\ -\phi_{10})^3, \\
w_4 &= (\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon \between \phi_{01},\ -\phi_{10})^4.
\end{aligned}
$$

We therefore require an algebraically complete aggregate of this set of invariants and covariants.

It is to be noticed that the argument ∇ does not appear in the equations $(I_2)''$, $(I_3)'$, $(I_4)'$; so that it is a solution of the equations, and it must be associated with the required algebraically complete aggregate of concomitants of the binary forms.

The three equations $(I_2)''$, $(I_3)'$, $(I_4)'$ constitute a complete Jacobian system, and the number of arguments which occur is 33; hence the algebraically complete aggregate of solutions contains 30 solutions, which thus give the algebraically complete aggregate of concomitants of the system of binary forms.

This aggregate is known* to be (or to be equivalent to) the following:—

the linear quantic, w_1;

the quadratic w_2, and its Hessian (discriminant) $EG - F^2$;

. . . . . w'_2, ... $LN - M^2$;

. . . . . w''_2, ... $ac - b^2$;

. . . . . w'''_2, ... $a'c' - b'^2$;

the cubic w_3, its Hessian, and either its discriminant or its cubicovariant;

the cubic w'_3, its Hessian, and either its discriminant or its cubicovariant;

the cubic w''_3, its Hessian, and either its discriminant or its cubicovariant;

the quartic w_4, its Hessian, its quadrinvariant, and its cubinvariant;

together with the Jacobians of any one of the forms w, say w_2, with all the rest of the forms. This makes up the requisite total of 30.

The asyzygetic aggregate is, of course, vastly more extensive; but for the present purpose it is only an algebraically independent aggregate that is wanted. Many modifications in the latter are possible: what is necessary to secure is that any modification does not interfere with the algebraical completeness of the aggregate. For instance, consider the set composed of

$$w_2, \quad EG - F^2, \quad w''_2, \quad ac - b^2, \quad J(w_2, w''_2),$$

where

$$4J(w_2, w''_2) = \frac{\partial w_2}{\partial \phi_{10}} \frac{\partial w''_2}{\partial \phi_{01}} - \frac{\partial w_2}{\partial \phi_{01}} \frac{\partial w''_2}{\partial \phi_{10}};$$

in the asyzygetic system, there is an intermediate invariant $Ec - 2Fb + Ga$; we have

$$J^2 = w_2 w''_2 (Ec - 2Fb + Ga) - w_2^2 (ac - b^2) - w''^2_2 (EG - F^2),$$

and therefore, in the algebraical aggregate, $Ec - 2Fb + Ga$ can be included when any other (such as $ac - b^2$) is excluded. Such a change would be desirable if differential invariants, linear in the quantities a, b, c, were required.

* See a memoir by the author, 'American Journal of Mathematics,' vol. 12 (1890), §§ 17, 22, 30.

21. Accordingly, we can take as an algebraically complete aggregate, containing the 31 necessary members, the set which follows :—

(i.) ∇,

(ii.) w_1,

(iii.) $J(w_1, w_2) = (E\psi_{01} - F\psi_{10})\phi_{01} - (F\psi_{01} - G\psi_{10})\phi_{10}$,

(iv.) w_2,

(v.) $H(w_2) = EG - F^2 = V^2$,

(vi.) w'_2,

(vii.) $J(w_2, w'_2) = (EM - FL)\phi_{01}^2 - (EN - GL)\phi_{01}\phi_{10} + (FN - GM)\phi_{10}^2$,

(viii.) $H(w'_2) = LN - M^2$, or $I(w_2, w'_2) = EN - 2FM + GL$,

(ix.) w''_2,

(x.) $J(w_2, w''_2) = (Eb - Fa)\phi_{01}^2 + \ldots$,

(xi.) $H(w''_2) = ac - b^2$, or $I(w_2, w''_2) = Ec - 2Fb + Ga$,

(xii.) w'''_2,

(xiii.) $J(w_2, w'''_2) = (Eb' - Fa')\phi_{01}^2 + \ldots$,

(xiv.) $H(w'''_2) = a'c' - b'^2$, or $I(w_2, w'''_2) = Ec' - 2Fb' + Ga'$,

(xv.) w_3,

(xvi.) $H(w_3) = (PR - Q^2)\phi_{01}^2 + \ldots$,

(xvii.) $\Phi(w_3) = (P^2S - 3PQR + 2Q^3)\phi_{01}^3 + \ldots$, or $\Delta(w_3)$,

(xviii.) $J(w_2, w_3) = (EQ - FP)\phi_{01}^3 + \ldots$,

(xix.) w'_3,

(xx.) $H(w'_3) = (km - l^2)\phi_{01}^2 + \ldots$,

(xxi.) $\Phi(w'_3) = (k^2n - 3klm + 2l^3)\phi_{01}^3 + \ldots$, or $\Delta(w'_3)$,

(xxii.) $J(w_2, w'_3) = (El - Fk)\phi_{01}^3 + \ldots$,

(xxiii.) w''_3,

(xxiv.) $H(w''_3) = (k'm' - l'^2)\phi_{01}^2 + \ldots$,

(xxv.) $\Phi(w''_3) = (k'^2n' - 3k'l'm' + 2l'^3)\phi_{01}^3 + \ldots$, or $\Delta(w''_3)$,

(xxvi.) $I(w_2, w''_3) = (El' - Fk')\phi_{01}^3 + \ldots$,

(xxvii.) w_4,

(xxviii.) $H(w_2) = (\alpha\gamma - \beta^2)\phi_{01}^2 + \ldots$,

(xxix.) $I(w_2) = \alpha\epsilon - 4\beta\delta + 3\gamma^2$,

(xxx.) $J(w_2) = \alpha\gamma\epsilon + 2\beta\gamma\delta - \alpha\delta^2 - \beta^2\epsilon - \gamma^3$,

(xxxi.) $J(w_2, w_4) = (E\beta - F\alpha)\phi_{01}^4 + \ldots$

22. It is still necessary to satisfy equation $(I_1)''$. This will be effected as follows :

when f involves ϕ_{10} and ϕ_{01}, it must be homogeneous in them, say of degree m_1; when f involves ψ_{10} and ψ_{01}, it must be homogeneous in them, say of degree m_2; when f involves E, F, G, it must be homogeneous in them, say of degree m_3; likewise for L, M, N, say of degree m_4; likewise for P, Q, R, S, say of degree m_5; for α, β, γ, δ, ϵ, say of degree m_6; for a, b, c, say of degree m_7; for a', b', c', say of degree m_8; for k, l, m, n, say of degree m_9; for k', l', m', n', say of degree m_{10}; and for ∇, of degree m_{11}; provided the value of μ, the index of the invariant, is given by

$$2\mu = m_1 + m_2 + 2(m_3 + m_4) + 3m_5 + 4m_6$$
$$+ 6m_7 + 6m_8 + 11(m_9 + m_{10}) + 8m_{11}.$$

This relation determines the indices of the whole system, as follows:—

Index = 1, w_1;

Index = 2, $\mathrm{J}(w_1, w_2)$, w_2, $\mathrm{H}(w_2)$, w'_2, $\mathrm{H}(w'_2)$ and $\mathrm{I}(w_2, w'_2)$;

Index = 3, $\mathrm{J}(w_2, w'_2)$, w_3;

Index = 4, ∇, w''_2, $\mathrm{I}(w_2, w''_2)$, w'''_2, $\mathrm{I}(w_2, w'''_2)$, $\mathrm{H}(w_3)$, $\mathrm{J}(w_2, w_3)$, w_4, $\mathrm{I}(w_4)$;

Index = 5, $\mathrm{J}(w_2, w''_2)$, $\mathrm{J}(w_2, w'''_2)$, $\mathrm{J}(w_2, w_4)$;

Index = 6, $\mathrm{H}(w''_2)$, $\mathrm{H}(w'''_2)$, $\Phi(w_3)$ and $\Delta(w_3)$, $\mathrm{H}(w_4)$, $\mathrm{J}(w_4)$;

Index = 7, w'_3, w''_3;

Index = 8, $\mathrm{J}(w_2, w'_3)$, $\mathrm{J}(w_2, w''_3)$;

Index = 12, $\mathrm{H}(w'_3)$, $\mathrm{H}(w''_3)$;

Index = 18, $\Phi(w'_3)$, $\Phi(w''_3)$;

Index = 22, $\Delta(w'_3)$, $\Delta(w''_3)$.

23. All these are relative invariants, that is to say, when the same function F of new variables is formed as the function f is of the old variables, then

$$\Omega^\mu \mathrm{F} = f,$$

where μ is the index of f, and $\Omega = \dfrac{\partial(\mathrm{X}, \mathrm{Y})}{\partial(x, y)}$. In order to have the absolute invariants, it is sufficient to divide each of them by a proper power of any one of them. For this purpose, we choose

$$\mathrm{V}^2 = \mathrm{H}(w_2) = \mathrm{EG} - \mathrm{F}^2;$$

we can regard V as of index unity, and therefore it will be sufficient to divide the relative invariants by a power of V equal to its index. We therefore have the set of 30 absolute invariants, given by

$$\frac{w_1}{V},\ \frac{J(w_1, w_2)}{V^2},\ \frac{w_2}{V^2},\ \frac{w'_2}{V^2},\ \frac{H(w'_2) \text{ and } I(w_2, w'_2)}{V^2},\ \frac{w''_2}{V^4},$$

$$\frac{H(w''_2)}{V^6} \text{ and } \frac{I(w_2, w''_2)}{V^4},\ \frac{w'''_2}{V^4},\ \frac{H(w'''_2)}{V^6} \text{ and } \frac{I(w_2, w'''_2)}{V^4},\ \frac{J(w_2, w'_2)}{V^3},$$

$$\frac{J(w_2, w''_2)}{V^5},\ \frac{J(w_2, w'''_2)}{V^5},\ \frac{w_3}{V^3},\ \frac{w'_3}{V^7},\ \frac{w''_3}{V^7},\ \frac{\nabla}{V^4},\ \frac{H(w_3)}{V^4},\ \frac{J(w_2, w_3)}{V^4},$$

$$\frac{H(w'_3)}{V^{12}},\ \frac{J(w_2, w'_3)}{V^8},\ \frac{H(w''_3)}{V^{12}},\ \frac{J(w_2, w''_3)}{V^8},\ \frac{w_4}{V^4},\ \frac{I(w_4)}{V^4},\ \frac{J(w_2, w_4)}{V^5},$$

$$\frac{\Phi(w_3) \text{ and } \Delta(w_3)}{V^6},\ \frac{\Phi(w'_3)}{V^{18}} \text{ and } \frac{\Delta(w'_3)}{V^{22}},\ \frac{\Phi(w''_3)}{V^{18}} \text{ and } \frac{V(w''_3)}{V^{22}},\ \frac{H(w_4)}{V^6},\ \frac{J(w_4)}{V^6}.$$

These thirty differential invariants constitute *the algebraically complete aggregate in terms of which all invariants*, involving (i.) some or all of the derivatives of the fundamental magnitudes E, F, G, L, M, N, up to the second order inclusive, as well as the magnitudes themselves, (ii.) the magnitudes P, Q, R, S, α, β, γ, δ, ϵ, (iii.) and the derivatives of two functions ϕ and ψ up to the third order inclusive, *can be expressed algebraically.* But it is to be noted that this inference is concerned solely with the partial differential equations, and it assumes that the various quantities E, F, G, L, M, N, and their derivatives are independent of one another; if any relations should subsist, owing to the intrinsic nature of the magnitudes, then the number of invariants in the above complete aggregate will be diminished by the number of relations.

Now one such relation is known; it is the relation commonly associated with GAUSS's name, and it expresses $LN - M^2$ in terms of E, F, G, and their derivatives up to the second order inclusive. But $LN - M^2$ is $H(w'_2)$ in the foregoing set; and, as will be seen later (§ 35) in the course of the geometrical interpretation, we have

$$\nabla = 4H(w_2)\, H(w'_2),$$

so that the number must be diminished by unity. Accordingly, *the algebraically complete aggregate of differential invariants, involving the magnitudes up to the specified order of derivation, contains* **29** *members; in terms of these members, every other invariant, involving the same magnitudes up to the specified order of derivation, can be expressed algebraically.*

24. As an illustration of the remark in § 6, we can obtain MINDING's expression for the geodesic curvature, quoted* by Professor ŽORAWSKI as an invariant. Let $\phi = 0$ be the equation of the curve, then

$$\phi_{10} + \phi_{01}y' = 0,$$

$$\phi_{20} + 2\phi_{11}y' + \phi_{02}y'^2 + \phi_{01}y'' = 0,$$

so that

$$-\phi_{01}{}^3 y'' = \phi_{20}\phi_{01}{}^2 - 2\phi_{11}\phi_{10}\phi_{01} + \phi_{02}\phi_{10}{}^2.$$

* *Loc. cit.*, p. 63.

Now $w''_2 V^{-4}$ is an invariant, as also is $w_2 V^{-2}$; hence

$$\frac{w''_2}{V w_2^{\frac{3}{2}}}$$

is an invariant, say U, so that

$$U = \frac{1}{V} \frac{a\phi_{01}^2 - 2b\phi_{01}\phi_{10} + c\phi_{10}^2}{(E\phi_{01}^2 - 2F\phi_{01}\phi_{10} + G\phi_{10}^2)^{\frac{3}{2}}}$$

$$= \frac{1}{V(E\phi_{01}^2 - 2F\phi_{01}\phi_{10} + G\phi_{10}^2)^{\frac{3}{2}}} [\{2V^2\phi_{20} + (E_{01} - 2F_{10}) r - E_{10}s\} \phi_{01}^2 - 2\{2V^2\phi_{11} - G_{10}r - E_{01}s\} \phi_{01}\phi_{10} + \{2V^2\phi_{02} - G_{01}r + (G_{10} - 2F_{01}) s\} \phi_{10}^2]$$

$$= \frac{1}{V(E + 2Fy' + Gy'^2)^{\frac{3}{2}}} [- 2V^2 y'' + (2GF_{01} - GG_{10} - FG_{01}) y'^3 + (2GE_{01} + 2FF_{01} - 3FG_{10} - EG_{01}) y'^2 - (2EG_{10} + 2FF_{10} - 3FE_{01} - GE_{10}) y' - (2EF_{10} - EE_{01} - FE_{10})]$$

$$= - \frac{2}{\rho''},$$

according to MINDING's expression for the geodesic curvature; or the geodesic curvature of the curve $\phi = 0$ is the invariant

$$- \tfrac{1}{2} \frac{w''_2}{V w_2^{\frac{3}{2}}}.$$

25. It is possible to make further inferences from the results. Thus we can settle the algebraically complete aggregate of invariants up to the order of derivatives retained, when those invariants are required which involve derivatives of E, F, G, and only one function, say ϕ. They manifestly constitute the aggregate, complete up to the order specified, of all the functions that remain invariant when the surface is deformed in any way without tearing or stretching, account being taken of a particular curve $\phi = 0$, and the invariance persisting through all changes of the independent variables of the surface. This aggregate, algebraically complete up to the order specified, consists of the nine members

$$\frac{w_2}{V^2}, \quad \frac{\nabla}{V^4},$$

$$\frac{w''_2}{V^4}, \quad \frac{H(w''_2)}{V^6} \text{ and } \frac{I(w_2, w''_2)}{V^4}, \quad \frac{J(w_2, w''_2)}{V^5},$$

$$\frac{w'_3}{V^7}, \quad \frac{H(w'_3)}{V^{12}}, \quad \frac{J(w_2, w'_3)}{V^8}, \quad \frac{\Phi(w'_3)}{V^{18}} \text{ and } \frac{\Delta(w'_3)}{V^{22}},$$

the first five of which were given by Professor ŽORAWSKI, who considered the specific aggregate only up to one order lower.

26. If we require the aggregate of invariants of this class involving derivatives of E, F, G up to order $n-1$ and derivatives of ϕ up to order n, the number of members in that algebraically complete aggregate can be obtained. *The total number of members is*

$$n^2;$$

it is composed of $\frac{1}{2}(n-1)(n-2)$ quantities which do not involve the derivatives of ϕ, these quantities being called Gaussian invariants of deformation, and their number having been determined* by ŽORAWSKI; and of $\frac{1}{2}n(n+3)-1$ quantities, each of which involves derivatives of ϕ. To make up the latter aggregate of $\frac{1}{2}n(n+3)-1$ quantities, we need (in addition to the binary forms already used) other binary forms of orders 4, 5, . . . , n; among these, the binary form of order m (for all values of m) has ϕ_{01} and $-\phi_{10}$ for its variables, and its coefficients are linear in the derivatives of ϕ up to order m inclusive; and the members, that would occur in the simplest expression of the aggregate through the existence of the binary form of order m, would be the quotients (by proper powers of V) of the binary form itself, of the $m-1$ (HERMITE'S) associated covariants, and of the Jacobian of w_2 and the binary form, making $m+1$ in all. Thus the total number† up to order n is

$$\begin{aligned} &1+3+4+\ldots+n \\ &=\tfrac{1}{2}n(n+3)-1, \end{aligned}$$

the number in question.

27. If we require the aggregate of differential invariants, which involve derivatives of E, F, G, L, M, N up to order $n-1$ and derivatives of a single function ϕ up to order n, the number in that algebraically complete aggregate can be obtained as follows. We can replace the derivatives of L, M, N of the specified orders by the introduction of the fundamental magnitudes of orders 3, 4, . . . , $n+1$ defined as the coefficients in the various powers of $\frac{dx}{ds}$ and $\frac{dy}{ds}$ in the complete expression of the quantities

$$\frac{d}{ds}\left(\frac{1}{\rho}\right),\quad \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right),\ldots,\ \frac{d^{n-1}}{ds^{n-1}}\left(\frac{1}{\rho}\right),$$

where ρ is the radius of curvature of the normal section through the tangent defined by $\frac{dx}{ds}$, $\frac{dy}{ds}$, and the arc-differentiation of $\frac{dx}{ds}$, $\frac{dy}{ds}$ is taken along the geodesic tangent‡.

When $n=2$, the system of binariants is composed of three quadratic forms with their three discriminants, a cubic form with its set of two associated covariants, and

* In his memoir, § 13.

† It will be noted that ∇V^{-4} in the aggregate in § 22 is a Gaussian invariant of deformation, and so is included among the $\frac{1}{2}(n-1)(n-2)$ quantities which do not involve ϕ.

‡ For the significance of this remark, see § 31, *post.*

the Jacobian of one of the quadratic forms with the other two quadratic forms and with the cubic form, being 12 in all. To include the next higher order given by $n = 3$, we need a cubic form with its set of two associated covariants, a quartic form with its set of three associated covariants, and the Jacobian of each of the forms with the originally selected quadratic form, being 9 in all. And so on in succession: the total number of binariants is

$$\begin{aligned} &12 + \{(9 + 11 + 13 + . \quad + (2n + 3)\} \\ &= n^2 + 4n. \end{aligned}$$

With these must be associated the $\frac{1}{2}(n-1)(n-2)$ quantities that do not involve the derivatives of ϕ, these being the Gaussian invariants of deformation; hence the total number is

$$\tfrac{3}{2}n^2 + \tfrac{5}{2}n + 1.$$

But these are relative invariants; each of them must be divided by the appropriate power of V so that, as one of them is V^2 and the quotient is unity, thus making the function no longer an invariant of the surface, the number of absolute invariants is

$$\begin{aligned} &\tfrac{3}{2}n^2 + \tfrac{5}{2}n \\ &= \tfrac{1}{2}n(3n+5). \end{aligned}$$

28. Lastly, if we require the aggregate of differential invariants which involve derivatives of E, F, G, L, M, N up to order $n - 1$, and derivatives of two functions ϕ, ψ up to order n, the number can be obtained in a similar manner. As in § 27, we replace the derivatives of L, M, N of the specified orders by the fundamental magnitudes of orders $3, 4, \ldots, n + 1$. The algebraically complete aggregate of relative invariants of the surface up to the orders specified is composed of two portions. The first includes the $\frac{1}{2}(n-1)(n-2)$ quantities which do not involve the derivatives of ϕ and ψ, these being the Gaussian invariants of deformation, as before. The second is the algebraically complete aggregate of the system of concomitants of a set of binary forms, each divided by a proper power of V in order to give rise to an absolute invariant of the surface. This set of binary forms contains

1 quantic of order 1,
4 quantics . . 2,
3 3,
3 4,
. . . .
3 . . . . . n,
1 quantic . . . $n + 1$,

being $3n$ in all. With them must be coupled (a) their (HERMITE's) associated covariants, the number of which is

$$1 \,.\, 0 + 4 \,.\, 1 + 3\{2 + 3 + . \quad + (n-1)\} + 1 \,.\, n$$

$= \frac{3}{2}n^2 - \frac{1}{2}n + 1$; and (*b*) the Jacobian of any one of the quantics with each of the rest, being $3n - 1$ in all. Thus the tale of the concomitants of the binary forms

$$= \tfrac{3}{2}n + (\tfrac{3}{2}n^2 - \tfrac{1}{2}n + 1) + 3n - 1$$
$$= \tfrac{3}{2}n^2 + \tfrac{11}{2}n.$$

But these are relative invariants; each of them must be divided by the appropriate power of V, so that, as one of them is V^2, and the quotient is unity, thus making the function no longer an invariant of the surface, the number of absolute invariants from this source is $\frac{3}{2}n^2 + \frac{11}{2}n - 1$. Thus the required aggregate of invariants of the kind specified up to order n is, in all, equal to

$$\tfrac{1}{2}(n-1)(n-2) + \tfrac{3}{2}n^2 + \tfrac{11}{2}n - 1$$
$$= 2n^2 + 4n.$$

29. But all these numbers are subject to diminution by as many units as there are algebraically independent relations among the invariants, which do not occur merely through algebraical forms, but arise through intrinsic relations associated with the general theory of surfaces. One such relation, being GAUSS's equation, has already (§ 23) been mentioned; so that the number $2n^2 + 4n$ would certainly be diminished by unity. It might happen that certain other combinations of the fundamental magnitudes of the various orders could be expressed in terms of E, F, G and their derivatives, the combinations being invariants of the set of binary forms, and the expressions in terms of E, F, G, and their derivatives being invariants of deformation. Each such relation would diminish the number $2n^2 + 4n$ by a single unit.

So far as I am aware, GAUSS's equation is the only relation of the type indicated which has already been established; but there is reason (§ 56) for surmising that other relations of that type do actually subsist.

PART II.

GEOMETRIC SIGNIFICANCE OF THE INVARIANTS.

30. The algebraically complete aggregate of the invariants of a given surface and of any two curves drawn upon it has been proved to be determinable by the development of LIE's method, as used by Professor ŽORAWSKI for the invariants of deformation. The actual determination of the members of those aggregates, which belong to the lowest orders, has been made. Each such invariant has a geometric significance,

and the significance of some of them is known; we proceed to consider this aspect of the invariants.

In dealing with binariants, several methods are possible. There is the symbolical method. There is the method dependent upon the use of canonical forms for the various functions; the complete expression of each binariant must be used through each operation; in the present instance, the canonical form would arise by taking ϕ and ψ as the independent variables on the surface.* There is the method that depends upon the characteristic property of binariants, by which the leading term alone, being sufficient to determine the binariant uniquely, is used to replace the binariant. The last of these methods will be used.

31. We denote by s an arc of the curve $\phi = 0$, so that d/ds implies differentiation along the curve; and we denote by d/dn differentiation in a direction on the surface perpendicular to the curve. Where no confusion will arise, we shall use x', x'', . . . in place of $\dfrac{dx}{ds}$, $\dfrac{d^2x}{ds^2}$, . . .; and so with quantities other than x.

In constructing the fundamental quantities of order higher than the second, a normal section through the tangent to ϕ is drawn; successive derivatives of the curvature of this section at the point are constructed, and the values of the second derivatives of x and y are those connected with the geodesic property at the point.† Accordingly, it is effectively the geodesic tangent to ϕ that is drawn; we shall denote by t an arc of this geodesic, so that d/dt implies differentiation along the geodesic. As the curve and the geodesic touch one another, we have

$$\frac{du}{ds} = \frac{du}{dt},$$

when the quantities relate to tangential properties only; but

$$\frac{du}{ds} - \frac{du}{dt}$$

is not zero when the quantities relate to contact of higher orders. Thus

$$\frac{dx}{ds} = \frac{dx}{dt}, \qquad \frac{dy}{ds} = \frac{dy}{dt};$$

but

$$\frac{d}{ds}\left(\frac{1}{\rho'}\right) - \frac{d}{dt}\left(\frac{1}{\rho'}\right),$$

where $\dfrac{1}{\rho'}$ is the circular curvature of the geodesic tangent, is not zero.

* This method is used by DARBOUX, 'Théorie générale des Surfaces,' vol. 3, p. 203.

† See the paper quoted in § 4.

The Independent Magnitudes connected with the Curve.

32. Various magnitudes connected with the curve $\phi = 0$ are required; we take

$\frac{1}{\rho}$ = its circular curvature,

$\frac{1}{\tau}$ = its curvature of torsion,

$\left.\begin{array}{l}\frac{1}{\rho'} = \text{the circular curvature} \\ \frac{1}{\tau'} = \text{the curvature of torsion}\end{array}\right\}$ of the geodesic tangent,

$\frac{1}{\rho''}$ = its geodesic curvature,

R = the radius of the osculating sphere,

ϖ = the angle between the normal to the surface and the principal normal of $\phi = 0$, and

$$\text{B} = \frac{d\phi}{dn},$$

where dn is the normal distance at the point of $\phi = 0$ from the curve $\phi + d\phi = 0$. Further, we write

$$\text{A} = \text{L}x'^2 + 2\text{M}x'y' + \text{N}y'^2,$$

$$\text{W} = \frac{1}{\text{V}}\begin{vmatrix} \text{E}x' + \text{F}y', & \text{F}x' + \text{G}y' \\ \text{L}x' + \text{M}y', & \text{M}x' + \text{N}y' \end{vmatrix},$$

$$\text{N} = \frac{1}{\text{V}}\begin{vmatrix} \text{E}x' + \text{F}y', & mx'^2 + 2m'x'y' + m''y'^2 + \text{E}x'' + \text{F}y'' \\ \text{F}x' + \text{G}y', & nx'^2 + 2n'x'y' + n''y'^2 + \text{F}x'' + \text{G}y'' \end{vmatrix},$$

with the customary notation for m, m', m'', n, n', n''; then $\text{A} = 0$ gives the asymptotic lines, $\text{W} = 0$ gives the lines of curvature, $\text{N} = 0$ gives geodesic lines. Moreover,

$$\text{W}^2 = \text{AH} - \text{A}^2 - \text{K},$$

where H and K are the mean curvature and the specific curvature of the surface at the point, viz.,

$$\text{H} = \frac{1}{\rho_1} + \frac{1}{\rho_2}, \qquad \text{K} = \frac{1}{\rho_1\rho_2},$$

ρ_1 and ρ_2 being the principal radii of curvature. We have the relations*

* See STAHL und KOMMERELL, 'Die Grundformeln der allgemeinen Flächentheorie,' § 14, for some of them.

$$\frac{1}{\rho'} = \frac{\cos \varpi}{\rho} = \mathrm{A},$$

$$\frac{1}{\rho''} = \frac{\sin \varpi}{\rho} = \mathrm{D},$$

$$\frac{1}{\tau} = \frac{d\varpi}{ds} - \mathrm{W},$$

$$\frac{1}{\tau'} = -\mathrm{W},$$

$$\mathrm{R}^2 = \rho^2 + \tau^2\left(\frac{d\rho}{ds}\right)^2,$$

$$\frac{d}{dt}\left(\frac{1}{\rho'}\right) = (\mathrm{P}, \mathrm{Q}, \mathrm{R}, \mathrm{S})(x', y')^3,$$

$$\frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right) = (\alpha, \beta, \gamma, \delta, \epsilon)(x', y')^4;$$

in the last two equations x' and y' are used in place of dx/dt and dy/dt, to which they are equal respectively. The relation

$$\mathrm{W}^2 = \mathrm{AH} - \mathrm{A}^2 - \mathrm{K}$$

at once gives

$$\frac{1}{\tau'^2} = \left(\frac{1}{\rho_1} - \frac{1}{\rho'}\right)\left(\frac{1}{\rho'} - \frac{1}{\rho_2}\right);$$

and we also have

$$\frac{1}{\tau} - \frac{1}{\tau'} = \frac{1}{\rho'^2 + \rho''^2}\left(\rho''\frac{d\rho'}{ds} - \rho'\frac{d\rho''}{ds}\right),$$

$$\frac{1}{\rho^2} = \frac{1}{\rho'^2} + \frac{1}{\rho''^2},$$

so that τ, ρ, and R are expressible in terms of ρ', ρ'', τ' and of their derivatives.

The Values of $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dx}{dn}, \frac{dy}{dn}$.

33. As regards $\frac{dx}{ds}\,(= x')$ and $\frac{dy}{ds}\,(= y')$, we have

$$\phi_{10}x' + \phi_{01}y' = 0,$$

$$\mathrm{E}x'^2 + 2\mathrm{F}x'y' + \mathrm{G}y'^2 = 1;$$

and therefore

$$x' = \frac{\phi_{01}}{\sqrt{w_2}}, \quad y' = -\frac{\phi_{10}}{\sqrt{w_2}}.$$

Next, differentiating along a direction in the surface that is perpendicular to the tangent to ϕ, we take the direction determined by dx/dn and dy/dn as being perpendicular to the direction determined by x' and y'; hence

$$Ex'\frac{dx}{dn} + F\left(y'\frac{dx}{dn} + x'\frac{dy}{dn}\right) + Gy'\frac{dy}{dn} = 0.$$

Moreover

$$E\left(\frac{dx}{dn}\right)^2 + 2F\frac{dx}{dn}\frac{dy}{dn} + G\left(\frac{dy}{dn}\right)^2 = 1;$$

and therefore

$$\frac{dx}{dn} = -\frac{1}{V}(Fx' + Gy') = \frac{1}{V\sqrt{w_2}}(-F\phi_{01} + G\phi_{10}),$$

$$\frac{dy}{dn} = \frac{1}{V}(Ex' + Fy') = \frac{1}{V\sqrt{w_2}}(E\phi_{01} - F\phi_{10}),$$

the quantities in the brackets in the last expressions being the quantities r and s of § 15.

Identification of the Simplest Invariants.

34. Using these results, we can at once obtain the interpretation of several of the invariants. We have

$$B = \frac{d\phi}{dn}$$

$$= \phi_{10}\frac{dx}{dn} + \phi_{01}\frac{dy}{dn}$$

$$= \frac{\sqrt{w_2}}{V},$$

and therefore

$$\frac{w_2}{V^2} = B^2.$$

Again,

$$A = (L, M, N)(x', y')^2$$

$$= \frac{1}{w_2}(L, M, N)(\phi_{01}, -\phi_{10})^2$$

$$= \frac{w'_2}{w_2};$$

and therefore by the relations in § 32, we have

$$\frac{w'_2}{V^2} = \frac{B^2}{\rho'}.$$

Also

$$W = \frac{1}{V}\{(EM - FL)x'^2 + \ldots\}$$

$$= \frac{1}{Vw_2}\{(EM - FL)\phi_{01}{}^2 + \ldots\}$$

$$= \frac{J(w_2, w'_2)}{Vw_2};$$

and therefore, by the relations in § 32, we have

$$\frac{J(w_2, w'_2)}{V^3} = -\frac{B^2}{\tau'}.$$

The result of § 24 gives

$$D = -\tfrac{1}{2}\frac{w''_2}{Vw_2^{\frac{3}{2}}},$$

and therefore, also by the relations in § 32, we have

$$\frac{w''_2}{V^4} = -2\frac{B^3}{\rho''}.$$

Also

$$\frac{d}{dt}\left(\frac{1}{\rho'}\right) = (P, Q, R, S)(x', y')^3$$

$$= \frac{w_3}{w_2^{\frac{3}{2}}},$$

so that

$$\frac{w_3}{V^3} = B^3\frac{d}{dt}\left(\frac{1}{\rho'}\right);$$

and

$$\frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right) = (\alpha, \beta, \gamma, \delta, \epsilon)(x', y')^4$$

$$= \frac{w_4}{w_2^{2}},$$

so that

$$\frac{w_4}{V^4} = B^4\frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right).$$

35. Certain invariants occur as belonging to the surface, independent of all curves such as $\phi = 0$. Of these, the most important is ∇V^{-4}; its value is given by

$$\frac{\nabla}{V^4} = \frac{4}{\rho_1\rho_2} = 4K.$$

But, as is well known, we also have

$$\frac{LN - M^2}{EG - F^2} = \frac{1}{\rho_1\rho_2} = K,$$

so that we have

$$\nabla = 4V^2 H(w'_2).$$

This is a relation among the differential invariants, and it is due to the intrinsic nature of the quantities E, F, G, L, M, N; accordingly, the number of algebraically independent invariants up to the present order must (§ 23) be diminished by unity, on account of the preceding relation.

It was noted, in § 21, that $H(w'_2)$ and $I(w_2, w'_2)$ are alternatives in a complete system, when w_2, $\mathrm{H}(w_2)$, w'_2, $\mathrm{J}(w_2, w'_2)$ are retained; as a matter of fact, the relation

$$\mathrm{J}^2(w_2, w'_2) = \mathrm{I}(w_2, w'_2)\, w_2 w'_2 - \mathrm{H}(w_2)\, w'^2_2 - \mathrm{H}(w'_2) w_2^2$$

subsists. Now the significance of $\mathrm{I}(w_2, w'_2)$ is known: we have

$$\frac{\mathrm{I}(w_2, w'_2)}{\mathrm{V}^2} = \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

Accordingly, we substitute the values that have been obtained, and we find

$$\frac{1}{\tau'^2} = \frac{1}{\rho'}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) - \frac{1}{\rho'^2} - \frac{1}{\rho_1\rho_2}$$
$$= \left(\frac{1}{\rho_1} - \frac{1}{\rho'}\right)\left(\frac{1}{\rho'} - \frac{1}{\rho_2}\right),$$

again the well-known relation giving the torsion of a geodesic at any point. This torsion vanishes when the geodesic is a tangent to a line of curvature.

Interpretation of the Remaining Invariants Associated with w_2, w'_2, w''_2, w_3.

36. We require the derivatives of w_2, w'_2, w''_2 with respect to the arc; for this purpose we shall use the property already quoted (§ 30)—that a binariant is uniquely determined by its leading term which, in the present instance, is the term involving the highest power of ϕ_{01}. Writing generally

$$u = f\phi^2_{01} - 2g\phi_{01}\phi_{10} + h\phi^2_{10},$$

we have

$$\frac{du}{ds} = 2f\phi_{01}(\phi_{11}x' + \phi_{02}y') - 2g\phi_{01}(\phi_{20}x' + \phi_{11}y') + \ldots$$
$$+ \phi_{01}^2(f_{10}x' + f_{01}y') + \ldots,$$

so that

$$\sqrt{w_2}\,\frac{du}{ds} = \phi_{01}^2(2f\phi_{11} - 2g\phi_{20}) + \ldots$$
$$+ \phi_{01}^3 f_{10} + \ldots$$
$$= \frac{1}{\mathrm{V}^2}\{(fb - ga)\,\phi_{01}^2 + \ldots\}$$
$$+ \frac{1}{\mathrm{V}^2}[\{f(\mathrm{EG}_{10} - \mathrm{FE}_{01}) + g(\mathrm{EE}_{01} - 2\mathrm{EF}_{10} + \mathrm{FE}_{10}) + \mathrm{V}^2 f_{10}\}\,\phi_{01}^3 + \ldots].$$

Firstly, let $f, g, h = \mathrm{E}, \mathrm{F}, \mathrm{G}$, so that u becomes w_2; then, on reduction, we find

$$\sqrt{w_2}\,\frac{dw_2}{ds} = \frac{\mathrm{J}(w_2, w''_2)}{\mathrm{V}^2} + \frac{w_2}{\mathrm{V}^2}\{(\mathrm{EG}_{10} - 2\mathrm{FF}_{10} + \mathrm{GE}_{10})\,\phi_{01} + \ldots\},$$

and consequently

$$\sqrt{w_2}\,\frac{d}{ds}\left(\frac{w_2}{V^2}\right) = \frac{J\,(w_2,\, w''_2)}{V^4}.$$

Secondly, let $f,\ g,\ h = L,\ M,\ N$, so that u becomes w'_2; then, on reduction, we find

$$\sqrt{w_2}\,\frac{dw'_2}{ds} = w_3 + \frac{1}{V^2 w_2}\{w'_2 J\,(w_2,\, w''_2) - w''_2 J\,(w_2,\, w'_2)\}$$
$$+ \frac{w'_2}{V^2}\{(EG_{10} - 2FF_{10} + GE_{10})\,\phi_{01} + \ldots\},$$

and consequently

$$\sqrt{w_2}\,\frac{d}{ds}\left(\frac{w'_2}{V^2}\right) = \frac{w_3}{V^2} + \frac{1}{V^2 w_2}\{w'_2 J\,(w_2,\, w''_2) - w''_2 J\,(w_2,\, w'_2)\}.$$

Thirdly, let $f,\ g,\ h = a,\ b,\ c$, so that u becomes w''_2; then, on reduction, we find

$$\sqrt{w_2}\,\frac{dw''_2}{ds} = \tfrac{1}{2}\,\frac{w'_3}{V^2} + \frac{2w''_2}{V^2}\{(EG_{10} - 2FF_{10} + GE_{10})\,\phi_{01} + \ldots\},$$

and consequently

$$\sqrt{w_2}\,\frac{d}{ds}\left(\frac{w''_2}{V^4}\right) = \tfrac{1}{2}\,\frac{w'_3}{V^6}.$$

The first of these gives

$$\frac{J\,(w_2,\, w''_2)}{V^5} = 2B^2\,\frac{dB}{ds},$$

and the third of them, taking account of the value of w''_2 which has already been obtained, gives

$$\frac{w'_3}{V^7} = -\,4B\,\frac{d}{ds}\left(\frac{B^3}{\rho''}\right).$$

The second of them can also be used to identify $J\,(w_2,\, w''_2)$, because all the other quantities occurring in the relation have been identified; the value is

$$\frac{J\,(w_2,\, w''_2)}{V^5} = B\rho'\,\frac{d}{ds}\left(\frac{B^2}{\rho'}\right) - B^3\rho'\left\{\frac{d}{dt}\left(\frac{1}{\rho'}\right) - \frac{2}{\rho''\tau'}\right\}.$$

Substituting the earlier value on the left-hand side, we have (after a slight reduction)

$$\frac{d}{dt}\left(\frac{1}{\rho'}\right) - \frac{d}{ds}\left(\frac{1}{\rho'}\right) = \frac{2}{\rho''\tau'},$$

being an illustration of the remark in § 31, and showing that in general the rate of change of the curvature of a normal section is not the same along the curve $\phi = 0$ and the geodesic, both of which touch that section. The result can also be written in the form

$$\frac{dA}{ds} = \Upsilon + 2DW,\quad \frac{dA}{dt} = \Upsilon,$$

with the earlier significance for A, D, W, and Υ is given by

$$\Upsilon = (\mathrm{P, Q, R, S} \between x', y')^3 ;$$

and another form is

$$\frac{d}{ds}\left(\frac{w'_2}{w_2}\right) = \frac{w_3}{w_2^{\frac{3}{2}}} - \frac{w''_2 \mathrm{J}(w_2, w'_2)}{\mathrm{V}^2 w_2^{\frac{3}{2}}}.$$

37. We require derivatives of some of the binary quadratics with respect to an arc in the surface normal to the curve $\phi = 0$; for this purpose, we proceed as in § 36. We take

$$u = f\phi_{01}^2 - 2g\phi_{01}\phi_{10} + h\phi_{10}^2,$$

and we have

$$\begin{aligned}\mathrm{V}\sqrt{w_2}\frac{du}{dn} = 2f\phi_{01}\{\phi_{11}(-\mathrm{F}\phi_{01} + \ldots) + \phi_{02}(\mathrm{E}\phi_{01} + \ldots)\} \\ - 2g\phi_{01}\{\phi_{20}(-\mathrm{F}\phi_{01} + \ldots) + \phi_{11}(\mathrm{E}\phi_{01} + \ldots)\} \\ + \phi_{01}^2\{f_{10}(-\mathrm{F}\phi_{01} + \ldots) + f_{01}(\mathrm{E}\phi_{01} + \ldots)\} + \ldots\ldots;\end{aligned}$$

and so, after some transformation and reduction, we find

$$\begin{aligned}\mathrm{V}^3\sqrt{w_2}\frac{du}{dn} = \phi_{01}^2\{f(\mathrm{E}c - \mathrm{F}b) - g(\mathrm{E}b - \mathrm{F}a)\} + \ldots \\ + \phi_{01}^3\{f\mathrm{E}(\mathrm{EG}_{01} + \mathrm{FG}_{10} - 2\mathrm{FF}_{01}) - (f\mathrm{F} + g\mathrm{E})(\mathrm{EG}_{10} - \mathrm{FE}_{01}) \\ + g\mathrm{F}(-\mathrm{EE}_{01} + 2\mathrm{EF}_{10} - \mathrm{FE}_{10}) + \mathrm{V}^2(-\mathrm{F}f_{10} + \mathrm{E}f_{01})\} + \ldots\ldots\end{aligned}$$

Firstly, let $f, g, h = \mathrm{E, F, G}$, so that u becomes w_2. The coefficient of the first term in the earlier aggregate is

$$= \mathrm{E}^2 c - 2\mathrm{EF}b + \mathrm{F}^2 a$$

$$= \mathrm{E}(\mathrm{E}c - 2\mathrm{F}b + \mathrm{G}a) - \mathrm{V}^2 a,$$

and therefore that aggregate is

$$= w_2 \mathrm{I}(w_2, w''_2) - \mathrm{V}^2 w''_2.$$

The coefficient of the first term in the later aggregate is

$$\mathrm{E}\{-\mathrm{F}(\mathrm{EG}_{10} - 2\mathrm{FF}_{10} + \mathrm{GE}_{10}) + \mathrm{E}(\mathrm{EG}_{01} - 2\mathrm{FF}_{01} + \mathrm{GE}_{01})\},$$

and therefore that aggregate is

$$= w_2 \mathrm{V}\sqrt{w_2}\frac{d\mathrm{V}^2}{dn}.$$

Consequently

$$\mathrm{V}^3\sqrt{w_2}\frac{dw_2}{dn} = w_2 \mathrm{I}(w_2, w''_2) - \mathrm{V}^2 w''_2 + w_2\mathrm{V}\sqrt{w_2}\frac{d\mathrm{V}^2}{dn},$$

and therefore

$$\mathrm{V}^5\sqrt{w_2}\frac{d}{dn}\left(\frac{w_2}{\mathrm{V}^2}\right) = w_2 \mathrm{I}(w_2, w''_2) - \mathrm{V}^2 w''_2.$$

Inserting the values of the invariants that are already known, we have

$$\mathrm{B} \,.\, 2\mathrm{B}\frac{d\mathrm{B}}{dn} = \mathrm{B}^2 \frac{\mathrm{I}(w_2, w''_2)}{\mathrm{V}^4} + 2\frac{\mathrm{B}^3}{\rho''},$$

and therefore

$$\frac{\mathrm{I}(w_2, w''_2)}{\mathrm{V}^4} = 2\left(\frac{d\mathrm{B}}{dn} - \frac{\mathrm{B}}{\rho''}\right).$$

Secondly, let $f, g, h = \mathrm{L, M, N}$, so that u becomes w'_2. Proceeding in the same way, we find

$$\mathrm{V}^5\sqrt{w_2}\,\frac{d}{dn}\left(\frac{w'_2}{\mathrm{V}^2}\right) = w'_2\mathrm{I}(w_2, w''_2) + \mathrm{V}^2\mathrm{J}(w_2, w_3) - \frac{1}{w_2}\{\mathrm{V}^2 w'_2 w''_2 + \mathrm{J}(w_2, w'_2)\,\mathrm{J}(w_2, w''_2)\}.$$

Inserting the values of those invariants which have already been obtained, we have (after a little reduction)

$$\frac{\mathrm{J}(w_2, w_3)}{\mathrm{V}^4} = \mathrm{B}^3\frac{d}{dn}\left(\frac{1}{\rho'}\right) - \frac{2\mathrm{B}^2}{\tau'}\frac{d\mathrm{B}}{ds}.$$

Thirdly, let $f, g, h = a, b, c$, so that u becomes w''_2. Proceeding in the same way as for w_2, we find

$$\mathrm{V}^7\sqrt{w_2}\,\frac{d}{dn}\left(\frac{w''_2}{\mathrm{V}^4}\right) = w_2\mathrm{H}(w''_2) - \tfrac{2}{3}w_2{}^2\nabla + \tfrac{1}{2}\mathrm{J}(w_2, w'_3).$$

Now we have retained $\mathrm{I}(w_2, w''_2)$ in our aggregate, in place of $\mathrm{H}(w''_2)$, so that the latter must be removed from the foregoing expression: as the relation

$$\mathrm{J}^2(w_2, w''_2) = w_2 w''_2\mathrm{I}(w_2, w''_2) - w_2{}^2\mathrm{H}(w''_2) - w''_2{}^2\mathrm{V}^2$$

holds, we have

$$\mathrm{V}^7 w_2{}^{\frac{3}{2}}\frac{d}{dn}\left(\frac{w''_2}{\mathrm{V}^4}\right) = \tfrac{1}{2}w_2\mathrm{J}(w_2, w'_3) - \tfrac{2}{3}w_2{}^3\nabla + w_2 w''_2\mathrm{I}(w_2, w''_2) - w''_2{}^2\mathrm{V}^2 - \mathrm{J}^2(w_2, w''_2).$$

Inserting the values of those invariants which have already been obtained, and reducing the equation, we ultimately have

$$\frac{\mathrm{J}(w_2, w'_3)}{\mathrm{V}^8} = -4\mathrm{B}^3\frac{d}{dn}\left(\frac{\mathrm{B}}{\rho''}\right) + 8\mathrm{B}^2\left(\frac{d\mathrm{B}}{ds}\right)^2 + \tfrac{16}{3}\mathrm{B}^4\mathrm{K}.$$

It may be noted, in passing, that the above equation, which gives the relation between $\mathrm{I}(w_2, w''_2)$ and $\mathrm{H}(w''_2)$, leads to the expression for $\mathrm{H}(w''_2)$ in the form

$$\frac{\mathrm{H}(w''_2)}{\mathrm{V}^6} = -4\frac{\mathrm{B}}{\rho''}\frac{d\mathrm{B}}{dn} - 4\left(\frac{d\mathrm{B}}{ds}\right)^2.$$

38. Again, it is known that

$$\left.\begin{aligned} \mathrm{V}^2\frac{\partial\mathrm{H}}{\partial x} &= \mathrm{GP} - 2\mathrm{FQ} + \mathrm{ER}, \qquad & \mathrm{V}^2\frac{\partial\mathrm{K}}{\partial x} &= \mathrm{NP} - 2\mathrm{MQ} + \mathrm{LR} \\ \mathrm{V}^2\frac{\partial\mathrm{H}}{\partial y} &= \mathrm{GQ} - 2\mathrm{FR} + \mathrm{ES}, \qquad & \mathrm{V}^2\frac{\partial\mathrm{K}}{\partial y} &= \mathrm{NQ} - 2\mathrm{MR} + \mathrm{LS} \end{aligned}\right\};$$

and therefore

$$\sqrt{w_2}\,V^2 \frac{dH}{ds} = (GP - 2FQ + ER)\,\phi_{01} + (GQ - 2FR + ES)(-\phi_{10})$$
$$= \mathfrak{a}_1,$$

say, where $\mathfrak{a}_1$ is a covariant of the system with index easily seen to be equal to 3. Now it is easy to verify that

$$(EQ - FP)^2 = (GP - 2FQ + ER)\,EP - (EG - F^2)\,P^2 - (PR - Q^2)\,E^2,$$

and therefore that

$$J^2(w_2, w_3) = w_2 w_3 \mathfrak{a}_1 - V^2 w_3^2 - w_2^2 H(w_3).$$

Consequently $\mathfrak{a}_1$ is expressible in terms of the members of the system; when the expression is substituted above, the result enables us to obtain the value of $H(w_3)\,V^{-4}$. But it is simpler to modify the original system of concomitants in § 21: we can replace $H(w_3)$ in that aggregate by $\mathfrak{a}_1$, and the modified aggregate still is complete. For the significance of $\mathfrak{a}_1$, we have

$$\frac{\mathfrak{a}_1}{V^3} = B\frac{dH}{ds}.$$

Further, we have

$$\sqrt{w_2}\,V^3 \frac{dH}{dn} = (GP - 2FQ + ER)(-F\phi_{01} + G\phi_{10})$$
$$+ (GQ - 2FR + ES)(E\phi_{01} - F\phi_{10})$$
$$= \mathfrak{a}_2,$$

say, where $\mathfrak{a}_2$ is a covariant of the system with index easily seen to be 4. It is easy to verify that

$$E^3(P^2S - 3PQR + 2Q^3) - EP^2\{E^2S - 3EFR + (EG + 2F^2)\,Q - FGP\}$$
$$= -3EP(EQ - FP)(GP - 2FQ + ER) + 2(EQ - FP)^3 + 2V^2P^2(EQ - FP),$$

and therefore

$$w_2^3\Phi(w_3) - w_2 w_3^2 \mathfrak{a}_2 + 3w_2 w_3 \mathfrak{a}_1 J(w_2, w_3) - 2J^3(w_2, w_3) - 2V^2 w_3^2 J(w_2, w_3) = 0.$$

Consequently $\mathfrak{a}_2$ is expressible in terms of members of the system; when the expression is substituted above, the result enables us to obtain the value of $\Phi(w_3)\,V^{-6}$. But, as in the last case, it is simpler to modify the original system of concomitants in § 21: we can replace $\Phi(w_3)$ in that aggregate by $\mathfrak{a}_2$, and the modified aggregate still is complete. For the significance of $\mathfrak{a}_2$, we have

$$\frac{\mathfrak{a}_2}{V^4} = B\frac{dH}{dn}.$$

An Aggregate for the Lowest Orders of Derivatives.

39. It may be remarked (and it is easy to verify the statement) that, if we desire an algebraically complete aggregate of invariants, involving derivatives of ϕ alone up to order 2 at the utmost, and derivatives of E, F, G up to order 1, and the fundamental magnitudes of the first three orders and no other quantities, such an aggregate is composed of

$$\frac{w_2}{V^2},\ \frac{w'_2}{V^2},\ \frac{H(w'_2)}{V^2} \text{ or } \frac{I(w_2, w'_2)}{V^2},\ \frac{w''_2}{V^4},\ \frac{H(w''_2)}{V^6} \text{ or } \frac{I(w_2, w''_2)}{V^4},$$

$$\frac{w_3}{V^3},\ \frac{H(w_3)}{V^4} \text{ or } \frac{\mathfrak{a}_1}{V^3},\ \frac{\Phi(w_3)}{V^6} \text{ or } \frac{\Delta(w_3)}{V^6} \text{ or } \frac{\mathfrak{a}_2}{V^4},\ \frac{J(w_2, w'_2)}{V^3},$$

$$\frac{J(w_2, w''_2)}{V^5}, \text{ and } \frac{J(w_2, w_3)}{V^4}.$$

Every other invariant of the surface involving only the same quantities that occur in these invariants can be expressed algebraically in terms of the members of this aggregate. The geometrical significance of each of the members has been obtained; if, therefore, the geometrical significance of any additional invariant is known, the algebraic equation expressing the invariant in terms of the retained aggregate will express a property of the surface and the curve. Such additional invariants are provided by $\frac{dK}{ds}$ and $\frac{dK}{dn}$; they should accordingly lead to properties of the surface and the curve.

Two New Relations.

40. We have

$$\sqrt{w_2}\, V^2 \frac{dK}{ds} = (NP - 2MQ + LR)\,\phi_{01} + (NQ - 2MR + LS)\,(-\phi_{10}).$$

But

$$(LQ - MP)^2 = (NP - 2MQ + LR)\,LP - (LN - M^2)\,P^2 - L^2(PR - Q^2),$$

and

$$LQ - MP = \frac{1}{E}\{L(EQ - FP) - P(EM - FL)\};$$

hence, taking account of the relation among the leading terms of the various concomitants, we have

$$\frac{\{w'_2 J(w_2, w_3) - w_3 J(w_2, w'_2)\}^2}{w_2^{\,2}} = \sqrt{w_2}\, w'_2 w_3 V^2 \frac{dK}{ds} - w_3^{\,2} H(w'_2) - w_2'^{\,2} H(w_3).$$

Consequently

$$\sqrt{w_2}\mathrm{V}^2\frac{d\mathrm{K}}{ds} = \frac{w'_2}{w_2{}^2w_3}\{\mathrm{J}^2(w_2, w_3) + w_2{}^2\mathrm{H}(w_3)\} - \frac{2}{w_2{}^2}\mathrm{J}(w_2, w_3)\,\mathrm{J}(w_2, w'_2)$$
$$+ \frac{w_3}{w_2{}^2w'_2}\{\mathrm{J}^2(w_2, w'_2) + w_2{}^2\mathrm{H}(w'_2)\}$$
$$= \frac{w'_2}{w_2{}^2w_3}(w_2w_3\mathfrak{a}_1 - \mathrm{V}^2w_3{}^2) - \frac{2}{w_2{}^2}\mathrm{J}(w_2, w_3)\,\mathrm{J}(w_2, w'_2)$$
$$+ \frac{w_3}{w_2{}^2w'_2}\{w_2w'_2\mathrm{I}(w_2, w'_2) - \mathrm{V}^2w'_2{}^2\},$$

so that $\frac{d\mathrm{K}}{ds}$ is expressed in terms of the members of the retained aggregate. Substituting the values of the invariants in the equation and dividing out by $\mathrm{V}^3\mathrm{B}$ after substitution, we find

$$\frac{d\mathrm{K}}{ds} - \frac{1}{\rho'}\frac{d\mathrm{H}}{ds} = \left(\mathrm{H} - \frac{2}{\rho'}\right)\frac{d}{dt}\left(\frac{1}{\rho'}\right) + \frac{2}{\tau'}\frac{d}{dn}\left(\frac{1}{\rho'}\right) - \frac{4}{\mathrm{B}\tau'^2}\frac{d\mathrm{B}}{ds},$$

a property which can be changed also into other forms, *e.g.*, by using the relation in § 36, which expresses $\frac{d}{dt}\left(\frac{1}{\rho'}\right)$ in terms of $\frac{d}{ds}\left(\frac{1}{\rho'}\right)$ and other magnitudes. Effecting this change, and substituting for $\frac{d\mathrm{K}}{ds}$, $\frac{d\mathrm{H}}{ds}$, $\frac{d}{ds}\left(\frac{1}{\rho'}\right)$, their values in terms of derivatives of ρ_1, ρ_2, ρ', we can express the relation in the form

$$-\frac{d}{ds}\left[\left(\frac{1}{\rho_1} - \frac{1}{\rho'}\right)\left(\frac{1}{\rho'} - \frac{1}{\rho_2}\right)\right] = \frac{2}{\tau'}\left\{\left(\frac{1}{\rho_1} + \frac{1}{\rho_2} - \frac{2}{\rho'}\right)\frac{1}{\rho''} + \frac{d}{dn}\left(\frac{1}{\rho'}\right)\right\} - \frac{4}{\mathrm{B}\tau'^2}\frac{d\mathrm{B}}{ds},$$

and therefore

$$\frac{1}{\tau'^2}\frac{d\tau'}{ds} = \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} - \frac{2}{\rho'}\right)\frac{1}{\rho''} + \frac{d}{dn}\left(\frac{1}{\rho'}\right) - \frac{2}{\mathrm{B}\tau'}\frac{d\mathrm{B}}{ds},$$

or, what is the same thing,

$$-\frac{d}{ds}\left(\frac{1}{\tau'}\right) = \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} - \frac{2}{\rho'}\right)\frac{1}{\rho''} + \frac{d}{dn}\left(\frac{1}{\rho'}\right) - \frac{2}{\mathrm{B}\tau'}\frac{d\mathrm{B}}{ds}$$
$$= \frac{d}{dn}\left(\frac{1}{\rho'}\right) + \left(\mathrm{H} - \frac{2}{\rho'}\right)\frac{1}{\rho''} - \frac{2}{\tau'\mathrm{B}}\frac{d\mathrm{B}}{ds}.$$

Proceeding to construct the other invariant that was suggested in § 39, we have

$$\sqrt{w_2}\mathrm{V}^3\frac{d\mathrm{K}}{dn} = (\mathrm{NP} - 2\mathrm{MQ} + \mathrm{LR})(-\mathrm{F}\phi_{01} + \mathrm{G}\phi_{10})$$
$$+ (\mathrm{NQ} - 2\mathrm{MR} + \mathrm{LS})(\mathrm{E}\phi_{01} - \mathrm{F}\phi_{10})$$

Let u denote the leading coefficient on the right-hand side, so that

$$u = \mathrm{SEL} - \mathrm{R}(2\mathrm{EM} + \mathrm{FL}) + \mathrm{Q}(\mathrm{EN} + 2\mathrm{FM}) - \mathrm{PFN};$$

and let α_1, α_2 denote the leading coefficients of $\mathfrak{a}_1$, $\mathfrak{a}_2$ respectively, so that

$$\alpha_1 = \text{GP} - 2\text{FQ} + \text{ER},$$
$$\alpha_2 = \text{E}^2\text{S} - 3\text{EFR} + (\text{EG} + 2\text{F}^2)\,\text{Q} - \text{FGP}.$$

Then it is not difficult to establish the identity

$$\text{E}u = \text{L}\alpha_2 - 2\,(\text{EM} - \text{FL})\,\alpha_1 + (\text{EN} - 2\text{FM} + \text{GL})\,(\text{EQ} - \text{FP})$$
$$- \frac{2}{\text{E}}\,(\text{EG} - \text{F}^2)\,\{\text{L}\,(\text{EQ} - \text{FP}) - \text{P}\,(\text{EM} - \text{FL})\}.$$

Noting that all the quantities on the right-hand side are leading coefficients of covariants, we change the identity into a relation among covariants; and the result, on division throughout by w_2, is

$$\sqrt{w_2}\,\text{V}^3\frac{d\text{K}}{dn} = \frac{w'_2}{w_2}\,\mathfrak{a}_2 - \frac{2}{w_2}\,\text{J}\,(w_2 w'_2)\,\mathfrak{a}_1 + \frac{1}{w_2}\,\text{I}\,(w_2 w'_2)\,\text{J}\,(w_2 w_3)$$
$$- 2\text{V}^2\frac{w'_2}{w_2{}^2}\,\text{J}\,(w_2,\, w_3) + 2\text{V}^2\frac{w_3}{w_2{}^2}\,\text{J}\,(w_2,\, w'_2),$$

so that $d\text{K}/dn$ is expressed in terms of the members of the retained aggregate. Substituting the values of the invariants in the equation and dividing out by V^4B after substitution, we find

$$\frac{d\text{K}}{dn} - \frac{1}{\rho'}\frac{d\text{H}}{dn} = \left(\text{H} - \frac{2}{\rho'}\right)\frac{d}{dn}\left(\frac{1}{\rho'}\right) - \frac{2}{\tau'}\frac{d}{dt}\left(\frac{1}{\rho'}\right) + \frac{4}{\text{B}\rho'\tau'}\frac{d\text{B}}{ds} + \frac{2}{\tau'}\frac{d\text{H}}{ds} - \frac{2\text{H}}{\text{B}\tau'}\frac{d\text{B}}{ds}.$$

Effecting the same transformation as before, by taking

$$\frac{d}{dn}\left(\text{K} - \frac{\text{H}}{\rho'} + \frac{1}{\rho'^2}\right) = \frac{d}{dn}\left(-\frac{1}{\tau'^2}\right),$$

we find

$$\frac{d}{dn}\left(\frac{1}{\tau'}\right) = \frac{d}{dt}\left(\frac{1}{\rho'}\right) - \frac{2}{\rho'\text{B}}\frac{d\text{B}}{ds} - \frac{d\text{H}}{ds} + \frac{\text{H}}{\text{B}}\frac{d\text{B}}{ds}$$
$$= \frac{d}{ds}\left(\frac{1}{\rho'}\right) + \frac{2}{\rho''\tau'} + \left(\text{H} - \frac{2}{\rho'}\right)\frac{1}{\text{B}}\frac{d\text{B}}{ds} - \frac{d\text{H}}{ds}.$$

Identification of the remaining Invariants obtained in § 23, with some Modifications of the System.

41. We proceed now to the identification of the invariants of the next higher order of derivatives; these involve derivatives of ϕ of the third order, derivatives of ψ of the third order, and the fundamental magnitudes of the fourth order. The method used is similar to that adopted in the preceding sections; we form derivatives, with

regard to s and to n, of the invariants already interpreted, identify the new forms by means of some member of the complete aggregate, and thus we obtain the interpretation of that member. Accordingly, we shall usually state the results without the calculations, the laborious character of which is greatly lightened by using the leading terms of the covariants only.

We have

$$\sqrt{w_2}\frac{d}{ds}\left(\frac{w_3}{V^3}\right) = \frac{w_4}{V^3} + \frac{3}{2V^5 w_2}\{w_3 J(w_2, w''_2) - w''_2 J(w_2, w_3)\}.$$

Inserting the values of the invariants which occur in this equation, and using the relation

$$\frac{d}{dt}\left(\frac{1}{\rho'}\right) = \frac{d}{ds}\left(\frac{1}{\rho'}\right) + \frac{2}{\rho''\tau},$$

obtained in § 36, we have

$$B\frac{d}{ds}\left\{B^3\frac{d}{ds}\left(\frac{1}{\rho'}\right) + \frac{2B^3}{\rho''\tau'}\right\}$$
$$= B^4\frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right) + 3B^2\frac{dB}{ds}\frac{d}{ds}\left(\frac{1}{\rho'}\right) + 3\frac{B^4}{\rho''}\frac{d}{dn}\left(\frac{1}{\rho'}\right),$$

and this easily leads to the relation

$$\frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right) - \frac{d^2}{ds^2}\left(\frac{1}{\rho'}\right) = \frac{2}{B^3}\frac{d}{ds}\left(\frac{B^3}{\rho''\tau'}\right) - \frac{3}{\rho''}\frac{d}{dn}\left(\frac{1}{\rho'}\right)$$
$$= 2\frac{d}{ds}\left(\frac{1}{\rho''\tau'}\right) + \frac{3}{\rho''}\frac{d}{ds}\left(\frac{1}{\tau'}\right) + \frac{3}{\rho''^2}\left(H - \frac{2}{\rho'}\right),$$

on using the expression for $\frac{d}{ds}\left(\frac{1}{\tau'}\right)$ obtained in § 40. The fact that the value of

$$\frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right) - \frac{d^2}{ds^2}\left(\frac{1}{\rho'}\right)$$

is different from zero is another illustration of the remark in § 36.

We also have

$$V^4\sqrt{w_2}\frac{d}{dn}\left(\frac{w_3}{V^3}\right) = J(w_2, w_4) - \tfrac{3}{2}\frac{H(w'_2)}{V^2}w_2 J(w_2, w'_2)$$
$$+ \frac{3}{2V^2}\left\{w_3 I(w_2, w''_2) - \frac{1}{w_2}J(w_2, w''_2)J(w_2, w_3) - V^2\frac{w''_2\, w_3}{w_2}\right\};$$

when we substitute for the respective invariants and reduce, we obtain an expression for $J(w_2, w_4)$ in the form

$$\frac{J(w_2, w_4)}{B^3V^5} = \frac{d^2}{dn\,ds}\left(\frac{1}{\rho'}\right) + 2\frac{d}{dn}\left(\frac{1}{\rho''\tau'}\right) - \tfrac{3}{2}\frac{K}{\tau'} + \frac{3}{B}\frac{dB}{ds}\left\{\frac{d}{dn}\left(\frac{1}{\rho'}\right) - \frac{2}{B\tau'}\frac{dB}{ds}\right\},$$

and the expression can be further modified by substituting the value of $\frac{1}{B}\frac{dB}{ds}$ given in § 40.

42. As the quantity H, the measure of the mean curvature of the surface at the point, has occurred in the invariants $\mathfrak{a}_1$ and $\mathfrak{a}_2$, and as the quantities $\frac{d^2H}{ds\,dn}$ and $\frac{d^2H}{dn\,ds}$ are not equal to one another, we construct the quantities

$$\frac{d}{ds}\left(\frac{\mathfrak{a}_1}{V^3}\right),\ \frac{d}{dn}\left(\frac{\mathfrak{a}_1}{V^3}\right),\ \frac{d}{ds}\left(\frac{\mathfrak{a}_2}{V^4}\right),\ \frac{d}{dn}\left(\frac{\mathfrak{a}_2}{V^4}\right).$$

It will appear that, by means of the second and the third of these, we can obtain the value of $\frac{d^2H}{ds\,dn} \backsim \frac{d^2H}{dn\,ds}$.

We have

$$\sqrt{w_2}\,\frac{d}{ds}\left(\frac{\mathfrak{a}_1}{V^3}\right) = \frac{1}{V^3}\{(E\gamma - 2F\beta + G\alpha)\,\phi_{01}{}^2 + \ldots\}$$
$$+ \tfrac{1}{2}\frac{H(w'_2)}{V^5}\{I(w_2, w'_2)\,w_2 - 2V^2 w'_2\}$$
$$+ \frac{1}{2V^5 w^2}\{\mathfrak{a}_1 J(w_2, w''_2) - \mathfrak{a}_2 w''_2\}.$$

Let

$$\mathfrak{h}_1 = (E\gamma - 2F\beta + G\alpha)\,\phi_{01}{}^2 + \ldots;$$

then as

$$(E\beta - F\alpha)^2 = E\alpha\,(E\gamma - 2F\beta + G\alpha) - (\alpha\gamma - \beta^2)\,E^2 - (EG - F^2)\,\alpha^2,$$

we have

$$J^2(w_2, w_4) = w_2\,w_4\mathfrak{h}_1 - w_2{}^2 H(w_4) - V^2 w_4{}^2.$$

Hence the invariant $\mathfrak{h}_1$ is expressible in terms of the members of the system; when the expression is substituted above, the result enables us to obtain the value of $H(w_4)\,V^{-6}$. But it is simpler to modify the original system of concomitants in § 21; we can replace $H(w_4)$ in the aggregate by $\mathfrak{h}_1$, and the modified aggregate is still complete. The index of $\mathfrak{h}_1$ is manifestly 4.

When the various values are substituted, we find

$$\frac{\mathfrak{h}_1}{V^4} = B^2\left\{\frac{d^2H}{ds^2} - \tfrac{1}{2}K\left(H - \frac{2}{\rho'}\right) - \frac{1}{\rho''}\frac{dH}{dn}\right\}.$$

43. We have

$$V\sqrt{w_2}\,\frac{d}{dn}\left(\frac{\mathfrak{a}_1}{V^3}\right) = \frac{1}{V^3}[\{E(G\beta - 2F\gamma + E\delta) - F(G\alpha - 2F\beta + E\gamma)\}\,\phi_{01}{}^2 + \ldots]$$
$$+ \frac{1}{2V^5 w_2}[\{w_2 I(w_2, w''_2) - V^2 w''_2\}\,\mathfrak{a}_1 - J(w_2, w''_2)\,\mathfrak{a}_2]$$
$$- \frac{H(w'_2)}{V^3}\,J(w_2, w'_2).$$

Let

$$\mathfrak{h}_2 = \{E(G\beta - 2F\gamma + E\delta) - F(G\alpha - 2F\beta + E\gamma)\}\,\phi_{01}^2 + \ldots;$$

then as

$$E^3(\alpha^2\delta - 3\alpha\beta\gamma + 2\beta^3) - E\alpha^2\{E^2\delta - 3EF\gamma + (EG + 2F^2)\beta - FG\alpha\}$$
$$= -3E\alpha(E\beta - F\alpha)(G\alpha - 2F\beta + E\gamma) + 2(E\beta - F\alpha)^3 + 2V^2\alpha^2(E\beta - F\alpha),$$

we have

$$w_2^3\Phi(w_4) - w_2w_4^2\mathfrak{h}_2 + 3w_2w_4\mathfrak{h}_1 J(w_2, w_4) - 2J^3(w_2, w_4) - 2V^2w_4^2 J(w_2w_4) = 0.$$

Consequently $\mathfrak{h}_2$ is expressible in terms of members of the system and of $\Phi(w_4)$; and $\Phi(w_4)$ is expressible in terms of w_4, $H(w_4)$, $I(w_4)$, $J(w_4)$. When the various expressions are substituted, we can modify the system of concomitants in § 21; we replace any one of them, say $I(w_4)$, by $\mathfrak{h}_2$, and the modified aggregate is still complete. The expression for $\frac{d}{dn}\left(\frac{\mathfrak{a}_1}{V^3}\right)$ then gives the significance of $\mathfrak{h}_2$, the index of which is 5; when the values of the invariants already interpreted are substituted, we find

$$\frac{\mathfrak{h}_2}{V^5} = B^2\left\{\frac{d^2H}{dn\,ds} - \frac{K}{\tau'} + \frac{1}{B}\frac{dB}{ds}\frac{dH}{dn}\right\}.$$

Similarly, we have

$$\sqrt{w_2}\,\frac{d}{ds}\left(\frac{\mathfrak{a}_2}{V^4}\right) = \frac{\mathfrak{h}_2}{V^4} - \frac{H(w'_2)}{V^4}J(w_2, w'_2) + \frac{1}{2V^2w_2}\{J(w_2, w''_2)\,\mathfrak{a}_2 + V^2w''_2\mathfrak{a}_1\};$$

and thus we obtain another expression for $\mathfrak{h}_2V^{-5}$ in the form

$$\frac{\mathfrak{h}_2}{V^5} = B^2\left\{\frac{d^2H}{ds\,dn} - \frac{K}{\tau'} + \frac{1}{\rho''}\frac{dH}{ds}\right\}.$$

Comparing the two values thus obtained for $\mathfrak{h}_2$, we at once have an expression for $\frac{d^2H}{ds\,dn} - \frac{d^2H}{dn\,ds}$, given by

$$\frac{d^2H}{ds\,dn} - \frac{d^2H}{dn\,ds} = -\frac{1}{\rho''}\frac{dH}{ds} + \frac{1}{B}\frac{dB}{ds}\frac{dH}{dn}.$$

44. We proceed in the same way with $\frac{d}{dn}\left(\frac{\mathfrak{a}_2}{V^4}\right)$. To simplify the system we introduce a covariant $\mathfrak{h}_3$ of index 6, defined as

$$\mathfrak{h}_3 = \{E^2(G\gamma - 2F\delta + E\epsilon) - 2EF(G\beta - 2F\gamma + E\delta) + F^2(G\alpha - 2F\beta + E\gamma)\}\,\phi_{01}^2 + \ldots;$$

this covariant is expressible in terms of w_4, $H(w_4)$, $I(w_4)$, $J(w_4)$, $J(w_2, w_4)$ and, as

H (w_4) and I (w_4) have already been replaced by $\mathfrak{h}_1$ and $\mathfrak{h}_2$, we replace J (w_4) by $\mathfrak{h}_3$, leaving the modified aggregate still complete. Then we have

$$V\sqrt{w_2}\,\frac{d}{dn}\left(\frac{\mathfrak{a}_2}{V^4}\right) = \frac{\mathfrak{h}_3}{V^4} + Kw'_2 - \tfrac{1}{2}HKw_2 + \frac{\mathfrak{a}_2}{2V^6}\,I\,(w_2 w''_2)$$
$$+ \frac{1}{2V^4 w_2}\{\mathfrak{a}_1 J\,(w_2, w''_2) - w''_2\mathfrak{a}_2\},$$

which, after substitution of the known invariants and some reduction, leads to an expression for $\mathfrak{h}_3$ in the form

$$\frac{\mathfrak{h}_3}{V^6} = B^2\left\{\frac{d^2H}{dn^2} + \tfrac{1}{2}K\left(H - \frac{2}{\rho'}\right) - \frac{1}{B}\frac{dB}{ds}\frac{dH}{ds}\right\},$$

giving also the value of $\dfrac{d^2H}{dn^2}$ as an invariant.

45. The expressions for $\dfrac{d^2H}{ds^2}, \dfrac{d^2H}{ds\,dn}, \dfrac{d^2H}{dn\,ds}, \dfrac{d^2H}{dn^2}$ can be obtained in another way; it will be sufficiently illustrated by constructing the first of them. From the expressions for $V^2\dfrac{\partial H}{\partial x}$, $V^2\dfrac{\partial H}{\partial y}$ in § 38, we find the following by differentiation:

$$V^2H_{20} - \tfrac{1}{2}\,\frac{LN - M^2}{V^2}\{E\,(EN - 2FM + GL) - 2V^2L\}$$
$$= G\alpha - 2F\beta + E\gamma + PG\Gamma + Q\,(G\Delta - 2F\Gamma) + R\,(E\Gamma - 2F\Delta) + ES\Delta,$$

$$V^2H_{11} - \tfrac{1}{2}\,\frac{LN - M^2}{V^2}\{F\,(EN - 2FM + GL) - 2V^2M\}$$
$$= G\beta - 2F\gamma + E\delta + PG\Gamma' + Q\,(G\Delta' - 2F\Gamma') + R\,(E\Gamma' - 2F\Delta') + ES\Delta',$$

$$V^2H_{02} - \tfrac{1}{2}\,\frac{LN - M^2}{V^2}\{G\,(EN - 2FM + GL) - 2V^2N\}$$
$$= G\gamma - 2F\delta + E\epsilon + PG\Gamma'' + Q\,(G\Delta'' - 2F\Gamma'') + R\,(E\Gamma'' - 2F\Delta'') + ES\Delta'',$$

where (§ 6)

$$\left.\begin{aligned} 2V^2\Gamma &= GE_{10} - F\,(2F_{10} - E_{01}) \\ 2V^2\Gamma' &= GE_{01} - FG_{10} \\ 2V^2\Gamma'' &= G\,(2F_{01} - G_{10}) - FG_{01} \end{aligned}\right\}, \qquad \left.\begin{aligned} 2V^2\Delta &= E\,(2F_{10} - E_{01}) - FE_{10} \\ 2V^2\Delta' &= EG_{10} - FE_{01} \\ 2V^2\Delta'' &= EG_{01} - F\,(2F_{01} - G_{10}) \end{aligned}\right\}.$$

Knowing the values of x' and y', we form $\dfrac{dx'}{dx}, \dfrac{dx'}{dy}, \dfrac{dy'}{dx}, \dfrac{dy'}{dy}$, and then we have

$$\frac{d^2x}{ds^2} = x'\frac{dx'}{dx} + y'\frac{dx'}{dy},$$

$$\frac{d^2y}{ds^2} = x'\frac{dy'}{dx} + y'\frac{dy'}{dy}.$$

The actual values are found to be

$$\frac{d^2x}{ds^2} = \frac{s}{w_2^2}(-\phi_{01}^2\phi_{20} + 2\phi_{01}\phi_{10}\phi_{11} - \phi_{10}^2\phi_{02})$$
$$-\tfrac{1}{2}\frac{\phi_{01}^2}{w_2^2}(E_{10}\phi_{01}^2 - 2F_{10}\phi_{01}\phi_{10} + G_{10}\phi_{10}^2)$$
$$+\tfrac{1}{2}\frac{\phi_{01}\phi_{10}}{w_2^2}(E_{01}\phi_{01}^2 - 2F_{01}\phi_{01}\phi_{10} + G_{01}\phi_{10}^2),$$

$$\frac{d^2y}{ds^2} = \frac{r}{w_2^2}(-\phi_{01}^2\phi_{20} + 2\phi_{01}\phi_{10}\phi_{11} - \phi_{10}^2\phi_{02})$$
$$+\tfrac{1}{2}\frac{\phi_{01}\phi_{10}}{w_2^2}(E_{10}\phi_{01}^2 - 2F_{10}\phi_{01}\phi_{10} + G_{10}\phi_{10}^2)$$
$$-\tfrac{1}{2}\frac{\phi_{10}^2}{w_2^2}(E_{01}\phi_{01}^2 - 2F_{01}\phi_{01}\phi_{10} + G_{01}\phi_{10}^2),$$

where r and s on the right-hand sides have the values given in § 15. Now

$$V^2\frac{d^2H}{ds^2} = V^2H_{20}x'^2 + 2V^2H_{11}x'y' + V^2H_{02}y'^2 + V^2H_{10}x'' + V^2H_{01}y'';$$

when we substitute the values of the various quantities and reduce, we have

$$\frac{d^2H}{ds^2} = \frac{1}{B^2}\frac{\mathfrak{h}_1}{V^4} + \tfrac{1}{2}K\left(H - \frac{2}{\rho'}\right) + \frac{1}{\rho''}\frac{dH}{dn},$$

the same value as before (§ 42).

Again, we know* that

$$\frac{dx}{dt} = x', \qquad \frac{dy}{dt} = y',$$

$$\frac{d^2x}{dt^2} = \Gamma x'^2 + 2\Gamma' x'y' + \Gamma'' y'^2,$$

$$\frac{d^2y}{dt^2} = \Delta x'^2 + 2\Delta' x'y' + \Delta'' y'^2.$$

Hence, as

$$\frac{d^2H}{dt^2} = H_{20}x'^2 + 2H_{11}x'y' + H_{02}y'^2 + H_{10}\frac{d^2x}{dt^2} + H_{01}\frac{d^2y}{dt^2},$$

we find, after substitution,

$$\frac{d^2H}{dt^2} = \frac{1}{B^2}\frac{\mathfrak{h}_1}{V^4} + \tfrac{1}{2}K\left(H - \frac{2}{\rho'}\right).$$

Consequently we have

$$\frac{d^2H}{ds^2} - \frac{d^2H}{dt^2} = \frac{1}{\rho''}\frac{dH}{dn},$$

* See the paper by the author, quoted in § 4.

another illustration of the remark in § 36. It may similarly be proved that

$$\frac{d^2\mathrm{K}}{ds^2} - \frac{d^2\mathrm{K}}{dt^2} = \frac{1}{\rho''}\frac{d\mathrm{K}}{dn}.$$

These are particular cases of the theorem, which can similarly be established: *If Ω denote any quantity, which is connected with any point on the surface and the expression of which is independent of the curve $\phi = 0$ through the point, then*

$$\frac{d^2\Omega}{ds^2} - \frac{d^2\Omega}{dt^2} = \frac{1}{\rho''}\frac{d\Omega}{dn}.$$

46. Proceeding to the identification of the two invariants $\mathrm{H}(w'_3)$ and $\Phi(w'_3)$, which involve the coefficients of w'_3, we construct $\frac{d}{ds}\mathrm{I}(w_2, w''_2)$ and $\frac{d}{dn}\mathrm{I}(w_2, w''_2)$, and we find

$$\sqrt{w_2}\frac{d}{ds}\left\{\frac{\mathrm{I}(w_2, w''_2)}{\mathrm{V}^4}\right\} = \frac{1}{2\mathrm{V}^6}\{(\mathrm{E}m - 2\mathrm{F}l + \mathrm{G}k)\phi_{01} + \ldots\},$$

$$\mathrm{V}\sqrt{w_2}\frac{d}{dn}\left\{\frac{\mathrm{I}(w_2, w''_2)}{\mathrm{V}^4}\right\} = \frac{1}{2\mathrm{V}^6}[\{\mathrm{E}(\mathrm{E}n - 2\mathrm{F}m + \mathrm{G}l) - \mathrm{F}(\mathrm{E}m - 2\mathrm{F}l + \mathrm{G}k)\}\phi_{01} + \ldots] - \tfrac{1}{3}w_2\frac{\nabla}{\mathrm{V}^4}.$$

Let these covariants be denoted by $\mathfrak{c}_1$, $\mathfrak{c}_2$, respectively, so that

$$\mathfrak{c}_1 = (\mathrm{E}m - 2\mathrm{F}l + \mathrm{G}k)\phi_{01} + \ldots$$

$$\mathfrak{c}_2 = \{\mathrm{E}(\mathrm{E}n - 2\mathrm{F}m + \mathrm{G}l) - \mathrm{F}(\mathrm{E}m - 2\mathrm{F}l + \mathrm{G}k)\}\phi_{01} + \ldots$$

Then $\mathfrak{c}_1$ can be used to replace $\mathrm{H}(w'_3)$ in the aggregate as $\mathfrak{a}_1$ replaced $\mathrm{H}(w_3)$, and $\mathfrak{c}_2$ can be used to replace $\Phi(w'_3)$ in the aggregate as $\mathfrak{a}_2$ replaced $\Phi(w_3)$, in each case without affecting the completeness of the aggregate. The index of $\mathfrak{c}_1$ is 7; that of $\mathfrak{c}_2$ is 8. Their significance is given by

$$\frac{\mathfrak{c}_1}{\mathrm{V}^7} = 4\mathrm{B}\frac{d^2\mathrm{B}}{ds\,dn} - 4\mathrm{B}\frac{d}{ds}\left(\frac{\mathrm{B}}{\rho''}\right),$$

and

$$\frac{\mathfrak{c}_2}{\mathrm{V}^8} = 4\mathrm{B}\frac{d^2\mathrm{B}}{dn^2} - 4\mathrm{B}\frac{d}{dn}\left(\frac{\mathrm{B}}{\rho''}\right) + \tfrac{8}{3}\mathrm{B}^2\mathrm{K}.$$

47. We have

$$\sqrt{w_2}\frac{d}{ds}\left\{\frac{\mathrm{J}(w_2, w'_2)}{\mathrm{V}^3}\right\} - \frac{\mathrm{J}(w_2, w_3)}{\mathrm{V}^3}$$

$$= \frac{1}{\mathrm{V}^3 w_2}\mathrm{J}(w_2, w'_2)\,\mathrm{J}(w_2, w''_2) - \frac{1}{2\mathrm{V}^3}w''_2\,\mathrm{I}(w_2, w'_2) + \frac{w'_2 w''_2}{\mathrm{V}w_2},$$

and

$$V\sqrt{w_2}\frac{d}{dn}\left\{\frac{J(w_2, w'_2)}{V^3}\right\} - \frac{w_2\mathfrak{a}_1 - w_3V^2}{V^3}$$

$$= \frac{1}{V^5}J(w_2, w'_2)\,I(w_2, w''_2) - \frac{1}{2V^5}J(w_2, w''_2)\,I(w_2, w'_2)$$

$$+ \frac{1}{V^3 w_2}\{w'_2 J(w_2, w''_2) - w''_2 J(w_2, w'_2)\}.$$

When substitution is made for the various invariants, and the reduction is effected, we find

$$-\frac{d}{ds}\left(\frac{1}{\tau'}\right) = \frac{d}{dn}\left(\frac{1}{\rho'}\right) + \left(H - \frac{2}{\rho'}\right)\frac{1}{\rho''} - \frac{2}{B}\frac{dB}{ds}\frac{1}{\tau'},$$

$$-\frac{d}{dn}\left(\frac{1}{\tau'}\right) = -\frac{d}{ds}\left(\frac{1}{\rho'}\right) - \frac{2}{\rho''\tau'} - \left(H - \frac{2}{\rho'}\right)\frac{1}{B}\frac{dB}{ds} + \frac{dH}{ds}$$

$$= -\frac{d}{dt}\left(\frac{1}{\rho'}\right) - \left(H - \frac{2}{\rho}\right)\frac{1}{B}\frac{dB}{ds} + \frac{dH}{ds},$$

which are relations obtained earlier (§ 40). They show that $\frac{d}{ds}\left(\frac{1}{\tau'}\right)$ and $\frac{d}{dn}\left(\frac{1}{\tau'}\right)$ can be expressed in terms of the other magnitudes.

We also have

$$V^3\sqrt{w_2}\frac{d}{dn}\left\{\frac{J(w_2, w''_2)}{V^5}\right\} = \frac{1}{2V^5}J(w_2, w''_2)\,I(w_2, w''_2) + \tfrac{1}{2}\frac{w_2\mathfrak{c}_1}{V^5} - \tfrac{1}{2}\frac{w'_3}{V^3}.$$

All the covariants that occur in this relation are known; when we substitute their values and reduce the resulting expression, we find

$$\frac{d^2B}{ds\,dn} - \frac{d^2B}{dn\,ds} = \frac{1}{B}\frac{dB}{ds}\frac{dB}{dn} - \frac{1}{\rho''}\frac{dB}{ds}.$$

This result, and the corresponding result obtained for H in § 43, are special cases of the theorem, which can be established by using the invariantive forms: *If* Ω *denote any quantity, which is connected with any point on the surface and the expression of which is independent* of the curve* $\phi = 0$ *through the point, then*

$$\frac{d^2\Omega}{ds\,dn} - \frac{d^2\Omega}{dn\,ds} = \frac{1}{B}\frac{dB}{ds}\frac{d\Omega}{dn} - \frac{1}{\rho''}\frac{d\Omega}{ds}.$$

48. Similarly

$$\sqrt{w_2}\frac{d}{ds}\left\{\frac{J(w_2, w''_2)}{V^5}\right\} = \frac{1}{V^7}\{\tfrac{1}{2}w''_2\,I(w_2, w''_2) - w_2H(w''_2)\} + \frac{J(w_2, w'_3)}{2V^7} + \frac{\nabla}{3V^7}w^2_2;$$

* The *value* of B is not independent of the curve; but B is one of the fundamental quantities for the expression of properties of the curve, and its *expression* is an irresoluble variable.

after substitution and reduction, we have

$$\frac{1}{\mathrm{B}}\frac{d^2\mathrm{B}}{ds^2} = 2\mathrm{K} - \frac{d}{dn}\left(\frac{1}{\rho''}\right) + \frac{1}{\rho''^2} + 2\left(\frac{1}{\mathrm{B}}\frac{d\mathrm{B}}{ds}\right)^2.$$

Also, we have

$$\sqrt{w_2}\,\frac{d}{ds}\left\{\frac{\mathrm{J}(w_2, w_3)}{\mathrm{V}^4}\right\} - \frac{\mathrm{J}(w_2, w_4)}{\mathrm{V}^4} - \tfrac{1}{2}\frac{\mathrm{K}w_2\mathrm{J}(w_2, w'_2)}{\mathrm{V}^4}$$
$$= \frac{1}{2\mathrm{V}^6}\left\{-2w''_2\mathfrak{a}_1 + \frac{3}{w_2}\mathrm{J}(w_2, w''_2)\,\mathrm{J}(w_2, w_3) + \mathrm{V}^2\frac{w''_2 w_3}{w_2}\right\};$$

thence a value of $\mathrm{J}(w_2, w_4)$ is obtained in the form

$$\frac{\mathrm{J}(w_2, w_4)}{\mathrm{B}^3\mathrm{V}^5} = \frac{d^2}{ds\,dn}\left(\frac{1}{\rho'}\right) + \tfrac{1}{2}\frac{\mathrm{K}}{\tau'} - \frac{2}{\rho''}\frac{d\mathrm{H}}{ds} + \frac{1}{\rho''}\frac{d}{dt}\left(\frac{1}{\rho'}\right) - \frac{2}{\tau'}\left(\frac{1}{\mathrm{B}}\frac{d\mathrm{B}}{ds}\right)^2$$
$$+ \frac{2}{\mathrm{B}}\frac{d\mathrm{B}}{ds}\left\{\frac{d}{dn}\left(\frac{1}{\rho'}\right) + \left(\mathrm{H} - \frac{2}{\rho'}\right)\frac{1}{\rho''}\right\}.$$

Comparing this value of $\mathrm{J}(w_2, w_4)$ with the value that was obtained in § 41, we find

$$\frac{d^2}{ds\,dn}\left(\frac{1}{\rho'}\right) - \frac{d^2}{dn\,ds}\left(\frac{1}{\rho'}\right)$$
$$= \frac{2}{\tau'}\frac{d}{dn}\left(\frac{1}{\rho''}\right) + \frac{1}{\mathrm{B}}\frac{d\mathrm{B}}{ds}\frac{d}{dn}\left(\frac{1}{\rho'}\right) + \frac{1}{\rho''}\frac{d}{dt}\left(\frac{1}{\rho'}\right) - 2\frac{\mathrm{K}}{\tau'} - \frac{4}{\tau'}\left(\frac{1}{\mathrm{B}}\frac{d\mathrm{B}}{ds}\right)^2.$$

Lastly, we have

$$\mathrm{V}\sqrt{w_2}\,\frac{d}{dn}\left\{\frac{\mathrm{J}(w_2, w_3)}{\mathrm{V}^4}\right\} - \frac{w_2\mathfrak{h}_1}{\mathrm{V}^4} + \frac{w_4}{\mathrm{V}^4} - \tfrac{1}{2}\frac{\mathrm{K}}{\mathrm{V}^4}\{2w_2w'_2\mathrm{V}^2 - \mathrm{I}(w_2, w'_2)\,w_2^2\}$$
$$= -\frac{\mathrm{J}(w_2, w''_2)}{\mathrm{V}^6}\mathfrak{a}_1 + \frac{3}{2\mathrm{V}^6}\mathrm{J}(w_2, w_3)\,\mathrm{I}(w_2, w''_2)$$
$$+ \frac{3}{2w_2\mathrm{V}^4}\{w_3\mathrm{J}(w_2, w''_2) - w''_2\mathrm{J}(w_2, w_3)\}.$$

When we substitute the values of the invariants in this expression and reduce the result, we find

$$\frac{d^2}{dn^2}\left(\frac{1}{\rho'}\right) = \frac{d^2\mathrm{H}}{ds^2} - \mathrm{K}\left(\mathrm{H} - \frac{2}{\rho''}\right) - \frac{1}{\rho''}\frac{d\mathrm{H}}{dn} - \frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right)$$
$$+ \frac{1}{\mathrm{B}}\left[\frac{2}{\tau'}\frac{d^2\mathrm{B}}{dn\,ds} - 4\frac{d\mathrm{B}}{ds}\frac{d\mathrm{H}}{ds} + 5\frac{d\mathrm{B}}{ds}\frac{d}{dt}\left(\frac{1}{\rho'}\right)\right]$$
$$+ \frac{2}{\mathrm{B}^2}\frac{d\mathrm{B}}{ds}\left\{\left(\mathrm{H} - \frac{2}{\rho'}\right)\frac{d\mathrm{B}}{ds} - \frac{1}{\tau'}\frac{d\mathrm{B}}{dn}\right\}.$$

It is to be noted, from the results obtained in this section, that $\dfrac{d^2\mathrm{B}}{ds^2}$ and $\dfrac{d^2}{dn^2}\left(\dfrac{1}{\rho'}\right)$ are expressed in terms of the other magnitudes retained; or, if we choose, we can regard the last relation as determining $\dfrac{d^2\mathrm{B}}{dn\,ds}$ in terms of the other magnitudes retained.

Invariants involving Derivatives of Two Functions.

49. Among the aggregate of invariants set out in § 23, there still remain nine as yet uninterpreted; but their expressions involve derivatives of the function ψ. Five out of these nine involve derivatives of no higher order than those which occur in the invariants interpreted in §§ 34–38. In order to obtain their interpretation, it is convenient to associate with them invariants which depend upon ψ alone and bear the same relation to ψ alone as some of those already interpreted bear to ϕ alone; and then the complete aggregate can be simplified by replacing some of the original forms by some of the associated forms.

For this purpose, let

$$W'''_2 = (a', b', c' \between \psi_{01}, -\psi_{10})^2,$$

$$W_2 = (E, F, G \between \psi_{01}, -\psi_{10})^2,$$

$$J(W_2, W'''_2) = (Eb' - Fa')\psi_{01}^2 - (Ec' - Ga')\psi_{01}\psi_{10} + (Fc' - Gb')\psi_{10}^2,$$

$$\Delta_1 = a'\phi_{01}\psi_{01} - b'(\phi_{01}\psi_{10} + \phi_{10}\psi_{10}) + c'\phi_{10}\psi_{10},$$

$$\Delta_2 = 2(Eb' - Fa')\phi_{01}\psi_{01} - (Ec' - Ga')(\phi_{01}\psi_{10} + \phi_{10}\psi_{01}) + 2(Fc' - Gb')\phi_{10}\psi_{10};$$

then we establish (and it is easy to verify) the equations

$$J^2(w_1, w_2) - w_2 W_2 + V^2 w_1^2 = 0,$$

$$w_2\Delta_1 - w'''_2 J(w_1, w_2) + w_1 J(w_2, w'''_2) = 0,$$

$$\Delta_1^2 - w'''_2 W'''_2 + w_1^2 H(w'''_2) = 0,$$

$$w_2\Delta_2 - 2J(w_2, w'''_2) J(w_1, w_2) + w_1 w_2 I(w_2, w'''_2) = 2V^2 w_1 w'''_2,$$

$$\Delta_2^2 - 4J(w_2, w'''_2) J(W_2, W'''_2) = w_1^2\{I^2(w_2, w'''_2) - 4V^2 H(w'''_2)\}.$$

The first of these equations shows that W_2 can be regarded as known; it is not an independent invariant but, if we wished, we could replace w_2 by W_2 in the complete aggregate without affecting the completeness. This change will not be made; we shall retain W_2 as a quantity convenient for other purposes and alternative to w_2 in the aggregate.

The second and the third equations, combined so as to eliminate Δ, show that W'''_2 can be regarded as known; it is not an independent invariant but, if we wished, we could replace w'''_2 by W'''_2 in the complete aggregate without affecting the completeness. This change will be made.

The fourth and the fifth equations, combined so as to eliminate Δ_2, show that $J(W_2, W'''_2)$ can be regarded as known; it is not an independent invariant but, if we wished, we could replace $J(w_2, w'''_2)$ by $J(W_2, W'''_2)$ in the complete aggregate without affecting the completeness. This change also will be made.

The five invariants that remained for interpretation were

$$\frac{w_1}{V}, \quad \frac{J(w_1, w_2)}{V^2}, \quad \frac{w'''_2}{V^4}, \quad \frac{J(w_2, w'''_2)}{V^5}, \quad \frac{I(w_2, w'''_2)}{V^4};$$

after the changes that have been made, the five are

$$\frac{w_1}{V}, \quad \frac{J(w_1, w_2)}{V^2}, \quad \frac{W'''_2}{V^4}, \quad \frac{J(W_2, W'''_2)}{V^5}, \quad \frac{I(w_2, w'''_2)}{V^4},$$

of which the last may also be written $I(W_2, W'''_2)V^{-4}$. The interpretation of the first two of these is easily obtained; for the interpretation of the remaining three, which involve derivatives of ψ but not of ϕ, the results of earlier interpretations can be used.

50. For the purpose of the interpretation, we need certain geometric properties of the curve $\psi = 0$. Let ds' denote an elementary arc along the curve, and dn' an element along the normal to the curve; and let

$$A = \frac{d\psi}{dn'}.$$

Further, let $\frac{1}{\rho'_\psi}$ denote the circular curvature of the geodesic tangent to $\psi = 0$, and $\frac{1}{\tau'_\psi}$ the curvature of torsion of that tangent; also, let $\frac{1}{\rho''_\psi}$ denote the geodesic curvature of $\psi = 0$. Then W_2, W'''_2, $I(W_2, W'''_2)$, $J(W_2, W'''_2)$, stand to $\psi = 0$ in precisely the same relation as w_2, w''_2, $I(w_2, w''_2)$, $J(w_2, w''_2)$ to $\phi = 0$; and therefore

$$\frac{W_2}{V^2} = A^2,$$

$$\frac{W'''_2}{V^4} = -\frac{2A^3}{\rho''_\psi},$$

$$\frac{I(W_2, W'''_2)}{V^4} = 2\left(\frac{dA}{dn'} - \frac{A}{\rho''_\psi}\right),$$

$$\frac{J(W_2, W'''_2)}{V^5} = 2A^2\frac{dA}{ds'}.$$

Moreover, we have

$$\frac{dx}{ds'} = \frac{\psi_{01}}{\sqrt{W_2}}, \quad \frac{dy}{ds'} = \frac{-\psi_{10}}{\sqrt{W_2}};$$

so that, if λ be the angle at which $\phi = 0$ and $\psi = 0$ intersect, we have

$$\cos\lambda = E\frac{dx}{ds}\frac{dx}{ds'} + F\left(\frac{dx}{ds}\frac{dy}{ds'} + \frac{dy}{ds}\frac{dx}{ds'}\right) + G\frac{dy}{ds}\frac{dy}{ds'}$$

$$= \frac{J(w_1, w_2)}{\sqrt{w_2 W_2}},$$

and therefore

$$\frac{J(w_1, w_2)}{V^2} = AB \cos \lambda.$$

Also

$$\sin \lambda = V\left(\frac{dx}{ds'}\frac{dy}{ds} - \frac{dy}{ds'}\frac{dx}{ds}\right)$$

$$= \frac{Vw_1}{\sqrt{w_2 W_2}},$$

and therefore

$$\frac{w_1}{V} = AB \sin \lambda.$$

We can regard the quotient of the last two invariants as giving the angle λ; and we can regard the sum of their squares as defining the magnitude A. Clearly

$$J^2(w_1, w_2) + V^2 w^2_1 = V^4 A^2 B^2$$
$$= w_2 W_2,$$

a relation already used; it may be further used to replace $J(w_1, w_2)$ by W_2.

51. The general theory shows that all other invariants, which involve derivatives of ϕ and ψ up to the second order inclusive, derivatives of E, F, G of the first order, and the fundamental magnitudes of the first three orders, can be expressed in terms of the aggregate already retained, composed of the eleven invariants selected in § 39 and the five just identified, viz.:—

$$\frac{w_1}{V},\ \frac{J(w_1, w_2)}{V^2} \text{ or } \frac{W_2}{V^2},\ \frac{W'''_2}{V^4},\ \frac{J(W_2, W'''_2)}{V^3},\ \frac{I(W_2, W'''_2)}{V^4}.$$

It is not without interest to illustrate the property by one or two simple examples.

Consider the circular curvature of the geodesic tangent to $\psi = 0$; after the result in § 34, it manifestly will be given by

$$\frac{W'_2}{V^2} = \frac{A^2}{\rho'_\psi};$$

according to the theory, it ought to be expressible in terms of the invariants retained. Take

$$\nabla_1 = L\phi_{01}\psi_{01} - M(\phi_{01}\psi_{10} + \phi_{10}\psi_{01}) + N\phi_{10}\psi_{10};$$

then we have the equations

$$\nabla_1^2 = w'_2 W'_2 - w_1^2 H(w'_2),$$

$$w_2\nabla_1 = w'_2 J(w_1, w_2) - w_1 J(w_2, w'_2);$$

and therefore

$$w_2^2\{w'_2 W'_2 - w_1^2 H(w'_2)\} = \{w'_2 J(w_1, w_2) - w_1 J(w_2, w'_2)\}^2.$$

When the geometric values of all the invariants are substituted, the preceding relation (after mere simplification and division throughout by $A^2B^6V^8$) becomes

$$\frac{1}{\rho'\rho'_\psi} = \frac{\sin^2\lambda}{\rho_1\rho_2} + \left(\frac{\cos\lambda}{\rho'} + \frac{\sin\lambda}{\tau'}\right)^2,$$

a relation which can be verified independently by means of EULER's theorem on the curvature of a normal section and of the expression in § 35 for the torsion of the geodesic tangent.

Similarly for the curvature of torsion of the geodesic tangent to $\psi = 0$; after the result in § 34, it manifestly will be given by

$$\frac{J(W_2, W'_2)}{V^3} = -\frac{A^2}{\tau'_\psi}.$$

According to the theory, it also ought to be expressible in terms of the invariants retained. Take

$$\Phi = 2(EM - FL)\phi_{01}\psi_{01} - (EN - GL)(\phi_{01}\psi_{10} + \phi_{10}\psi_{01}) + 2(FN - GM)\phi_{10}\psi_{10};$$

then we have the equations

$$\Phi^2 = 4J(w_2, w'_2)\,J(W_2, W'_2) + w_1{}^2V^4\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^2,$$

$$w_2\Phi = 2J(w_1, w_2)\,J(w_2, w'_2) - w_1w_2I(w_2, w'_2) + 2V^2w_1w'_2,$$

and therefore

$$w_2{}^2\left\{4J(w_2, w'_2)\,J(W_2, W'_2) + w_1{}^2V^4\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^2\right\}$$
$$= \{2J(w_1, w_2)\,J(w_2, w'_2) - w_1w_2I(w_2, w'_2) + 2V^2w_1w'_2\}^2,$$

which gives an expression in terms of the invariants. When we substitute the values of all the invariants and divide out by $A^2B^6V^{10}$, we find

$$\frac{4}{\tau'\tau'_\psi} = -\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^2\sin^2\lambda + \left\{\frac{2\cos\lambda}{\tau'} - \frac{2\sin\lambda}{\rho'} + \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\sin\lambda\right\}^2.$$

That some results of this kind, connecting ρ' and ρ'_ψ, should exist, can easily be seen. When ρ_1 and ρ_2 are given, ρ' is determined by the inclination of $\phi = 0$ to a line of curvature; λ being given, we then know the inclination of $\psi = 0$ to that line of curvature, and so ρ'_ψ is known. Similarly for some result connecting τ' and τ'_ψ.

As a last illustration of this kind, consider the invariantive expressions for $\frac{dH}{ds'}$ and $\frac{dH}{dn'}$. Let $\mathfrak{b}_1$ and $\mathfrak{b}_2$ be the invariants corresponding to $\mathfrak{a}_1$ and $\mathfrak{a}_2$, so that

$$\mathfrak{b}_1 = (\mathrm{GP} - 2\mathrm{FQ} + \mathrm{ER})\,\psi_{01} + (\mathrm{GQ} - 2\mathrm{FR} + \mathrm{ES})\,(-\,\psi_{10}),$$

$$\mathfrak{b}_2 = (\mathrm{GP} - 2\mathrm{FQ} + \mathrm{ER})\,(-\,\mathrm{F}\psi_{01} + \mathrm{G}\psi_{10}) + (\mathrm{GQ} - 2\mathrm{FR} + \mathrm{ES})\,(\mathrm{E}\psi_{01} - \mathrm{F}\psi_{10})\,;$$

then

$$\frac{\mathfrak{b}_1}{\mathrm{V}^3} = \mathrm{A}\,\frac{d\mathrm{H}}{ds'}\,, \qquad \frac{\mathfrak{b}_2}{\mathrm{V}^4} = \mathrm{A}\,\frac{d\mathrm{H}}{dn'}\,.$$

Now we have the equations

$$\left.\begin{aligned} w_2\mathfrak{b}_1 &= \mathrm{J}\,(w_1, w_2)\,\mathfrak{a}_1 - w_1\mathfrak{a}_2 \\ w_2\mathfrak{b}_2 &= \mathrm{V}^2 w_1\mathfrak{a}_1 + \mathrm{J}\,(w_1, w_2)\,\mathfrak{a}_2 \end{aligned}\right\},$$

which are easily established; substituting in them the values of the invariants that occur, we find (on removing a factor $\mathrm{AB}^2\mathrm{V}^5$), the relations

$$\left.\begin{aligned} \frac{d\mathrm{H}}{ds'} &= \frac{d\mathrm{H}}{ds}\cos\lambda - \frac{d\mathrm{H}}{dn}\sin\lambda \\ \frac{d\mathrm{H}}{dn'} &= \frac{d\mathrm{H}}{ds}\sin\lambda + \frac{d\mathrm{H}}{dn}\cos\lambda \end{aligned}\right\},$$

which are the ordinary differential relations for transference from directions* ds and dn to ds' and dn', when the subject of operation is a function of position only and involves no properties of tangency and no properties of contact of order higher than the first. But for a function of position (and, *a fortiori*, for a function which involves properties of contact of the first order or of higher orders), the operators $\dfrac{d}{ds}$ and $\dfrac{d}{dn}$ are not interchangeable. Thus, in particular, $\dfrac{d^2\mathrm{H}}{ds\,dn}$ and $\dfrac{d^2\mathrm{H}}{dn\,ds}$ are not equal to one another, except for special curves; an expression for their difference has already been obtained.

52. It still remains to identify the four invariants w''_3, $\mathrm{H}\,(w''_3)$, $\Phi\,(w''_3)$, $\mathrm{J}\,(w_2, w''_3)$, which involve the derivatives of both ϕ and ψ. Instead of proceeding to obtain their values, we use the method adopted in § 49; we replace them by four equivalent invariants involving derivatives of ψ only, and the change does not affect the completeness of the aggregate. These four invariants are

$$\mathrm{W}''_3 = (k', l', m', n' \between \psi_{01}, -\,\psi_{10})^3,$$

$$\mathrm{H}\,(\mathrm{W}''_3) = (k'm' - l'^2)\,\psi_{01}{}^2 + \ldots$$

$$\Phi\,(\mathrm{W}''_3) = (k'^2n' - 3k'l'm' + 2l'^3)\,\psi_{01}{}^3 + \ldots,$$

$$\mathrm{J}\,(\mathrm{W}_2, \mathrm{W}''_3) = (\mathrm{E}l' - \mathrm{F}k')\,\psi_{01}{}^3 + \ldots$$

* The value of $\sin\lambda$ shows that the direction of dn' falls within the angle between ds and dn.

We then modify this set of four, and replace $H(W''_3)$ and $\Phi(W''_3)$ by $\mathfrak{E}_1$ and $\mathfrak{E}_2$ where

$$\mathfrak{E}_1 = (Em' - 2Fl' + Gk')\psi_{01} + \dots ,$$
$$\mathfrak{E}_2 = \{E(En' - 2Fm' + Gl') - F(Em - 2Fl + Gk)\}\psi_{01} + \dots ;$$

and the set W''_3, $J(W_2, W''_3)$, $\mathfrak{E}_1$, $\mathfrak{E}_2$ replace w''_3, $H(w''_3)$, $\Phi(w''_3)$, $J(w_2, w''_3)$ in the aggregate, which remains complete after the change. The set of equations, which exhibit the equivalence of the four inserted forms to the four ejected forms, is similar to the corresponding set in § 42; it is more complicated because the ground-forms w_3'', W''_3 are of the third order.

The geometric significance of the four inserted forms can be obtained from the consideration that they stand related to the curve $\psi = 0$ exactly as w'_3, $J(w_2, w'_3)$, $\mathfrak{e}_1$, $\mathfrak{e}_2$ to the curve $\phi = 0$. Adopting the notation of § 51, we thus have

$$\frac{W''_3}{V^7} = -4A\frac{d}{ds'}\left(\frac{A^3}{\rho''_\psi}\right),$$
$$\frac{J(W_2, W''_3)}{V^8} = -4A^3\frac{d}{dn'}\left(\frac{A}{\rho''_\psi}\right) + 8A^2\left(\frac{dA}{ds'}\right)^2 + \tfrac{16}{3}A^4K,$$
$$\frac{\mathfrak{E}_1}{V^7} = 4A\frac{d^2A}{ds'dn'} - 4A\frac{d}{ds}\left(\frac{A}{\rho''_\psi}\right),$$
$$\frac{\mathfrak{E}_2}{V^8} = 4A\frac{d^2A}{dn'^2} - 4A\frac{d}{dn'}\left(\frac{A}{\rho''_\psi}\right) + \tfrac{8}{3}A^2K.$$

All other properties of the curve $\psi = 0$ up to the order retained can be expressed in terms of the invariants of the aggregate; the examples given in § 51 will be a sufficient illustration of the remark.

The Aggregate for a Single Curve $\phi = 0$ up to the Order Retained.

53. The 29 invariants in the preceding set have a closer affinity to the curve $\phi = 0$ than to the curve $\psi = 0$, the chief reason being that the first derivatives of ϕ were made the variables for the binary forms. By taking the first derivatives of ψ for these variables an equivalent set of 29 invariants could be obtained, having a closer affinity to the curve $\psi = 0$ than to the curve $\phi = 0$. And it would be possible to modify each of these two sets, so as to construct a new equivalent set of 29, symmetrically related to the two curves. All that is necessary in each modification is to secure that the retained aggregate remains algebraically complete.

Out of the set of 29 invariants retained, there are 20 which are not affected by the curve $\psi = 0$ in their expression; and therefore we infer that all the differential invariants of a surface and a curve $\phi = 0$ upon the surface, involving derivatives of ϕ up to the third order inclusive, involving the magnitudes E, F, G and their derivatives up to the second order inclusive, involving also the fundamental magnitudes of the second, the third, and the fourth orders, can be expressed algebraically in terms of an algebraically complete aggregate of 20 members.

This aggregate is composed of 20 quantities, each divided by an appropriate power of V; the sections quoted give the significance of the respective invariants. These quantities are as follows:—

[§ 34] $$w_2 = (\mathrm{E},\ \mathrm{F},\ \mathrm{G} \between \phi_{01},\ -\phi_{10})^2,$$

[§ 34] $$w'_2 = (\mathrm{L},\ \mathrm{M},\ \mathrm{N} \between \phi_{01},\ -\phi_{10})^2,$$

[§ 34] $$\mathrm{J}(w_2, w'_2) = \begin{array}{|c|c|c|} \hline \mathrm{EM} & \mathrm{EN} & \mathrm{FN} \\ -\mathrm{FL} & -\mathrm{GL} & -\mathrm{GM} \\ \hline \end{array} \between \phi_{01},\ -\phi_{10})^2,$$

[§ 35] $$\mathrm{I}(w_2, w'_2) = \mathrm{EN} - 2\mathrm{FM} + \mathrm{GL},$$

[§ 34] $$w''_2 = (a,\ b,\ c \between \phi_{01},\ -\phi_{10})^2,$$

[§ 36] $$\mathrm{J}(w_2, w''_2) = \begin{array}{|c|c|c|} \hline \mathrm{E}b & \mathrm{E}c & \mathrm{F}c \\ -\mathrm{F}a & -\mathrm{G}a & -\mathrm{G}b \\ \hline \end{array} \between \phi_{01},\ -\phi_{10})^2,$$

[§ 37] $$\mathrm{I}(w_2, w''_2) = \mathrm{E}c - 2\mathrm{F}b + \mathrm{G}a,$$

[§ 34] $$w_3 = (\mathrm{P},\ \mathrm{Q},\ \mathrm{R},\ \mathrm{S} \between \phi_{01},\ -\phi_{10})^3,$$

[§ 37] $$\mathrm{J}(w_2, w_3) = \begin{array}{|c|c|c|c|} \hline \mathrm{EQ} & 2\mathrm{ER} & \mathrm{ES} & \\ -\mathrm{FP} & -\mathrm{FQ} & +\mathrm{FR} & \mathrm{FS} \\ & -\mathrm{GP} & -2\mathrm{GQ} & -\mathrm{GR} \\ \hline \end{array} \between \phi_{01},\ -\phi_{10})^3,$$

[§ 38] $$\mathfrak{a}_1 = \begin{array}{|c|c|} \hline \mathrm{ER} & \mathrm{ES} \\ -2\mathrm{FQ} & -2\mathrm{FR} \\ +\mathrm{GP} & +\mathrm{GQ} \\ \hline \end{array} \between \phi_{01},\ -\phi_{10}),$$

[§ 38] $$\mathfrak{a}_2 = \begin{array}{|c|c|} \hline \mathrm{E}^2\mathrm{S} & \mathrm{EFS} \\ -3\mathrm{EFR} & -(\mathrm{EG}+2\mathrm{F}^2)\mathrm{R} \\ +(\mathrm{EG}+2\mathrm{F}^2)\mathrm{Q} & +3\mathrm{FGQ} \\ -\mathrm{FGP} & -\mathrm{G}^2\mathrm{P} \\ \hline \end{array} \between \phi_{01},\ -\phi_{10}),$$

[§ 36] $$w'_3 = (k,\ l,\ m,\ n \between \phi_{01},\ -\phi_{10})^3,$$

[§ 37] $$J(w_2, w'_3) = \left|\begin{array}{c|c|c|c} El & 2Em & En & \\ -Fk & -Fl & +Fm & Fn \\ & -Gk & -2Gl & -Gm \end{array}\right|)\!(\phi_{01}, -\phi_{10})^3,$$

[§ 46] $$\mathfrak{e}_1 = \left|\begin{array}{c|c} Em & En \\ -2Fl & -2Fm \\ +Gk & +Gl \end{array}\right|)\!(\phi_{01}, -\phi_{10}),$$

[§ 46] $$\mathfrak{e}_2 = \left|\begin{array}{c|c} E^2n & EFn \\ -3EFm & -(EG+2F^2)m \\ +(EG+2F^2)l & +3FGl \\ -FGk & -G^2k \end{array}\right|)\!(\phi_{01}, -\phi_{10}),$$

[§ 34] $$w_4 = (\alpha, \beta, \gamma, \delta, \epsilon)\!(\phi_{01}, -\phi_{10})^4,$$

[§ 41] $$J(w_2, w_4) = \left|\begin{array}{c|c|c|c|c} E\beta & 3E\gamma & 3E\delta & E\epsilon & \\ -F\alpha & -2F\beta & & +2F\delta & F\epsilon \\ & -G\alpha & -3G\beta & -3G\gamma & -G\delta \end{array}\right|)\!(\phi_{01}, -\phi_{10})^4,$$

[§ 42] $$\mathfrak{h}_1 = \left|\begin{array}{c|c|c} E\gamma & E\delta & E\epsilon \\ -2F\beta & -2F\gamma & -2F\delta \\ +G\alpha & +G\beta & +G\gamma \end{array}\right|)\!(\phi_{01}, -\phi_{10})^2,$$

[§ 43] $$\mathfrak{h}_2 = \left|\begin{array}{c|c|c} E^2\delta & E^2\epsilon & EF\epsilon \\ -3EF\gamma & -2EF\delta & -(EG+2F^2)\delta \\ +(EG+2F^2)\beta & -2FG\beta & +3FG\gamma \\ -FG\alpha & -G^2\alpha & -G^2\beta \end{array}\right|)\!(\phi_{01}, -\phi_{10})^2,$$

[§ 44] $$\mathfrak{h}_3 = \left|\begin{array}{c|c|c} E^3\epsilon & E^2F\epsilon & EF^2\epsilon \\ -4E^2F\delta & -4EF^2\delta & -(2F^3+2EFG)\delta \\ +(E^2G+5EF^2)\gamma & +(2EFG+4F^3)\gamma & +(5F^2G+EG^2)\gamma \\ -(2EFG+2F^3)\beta & -4F^2G\beta & -4FG^2\beta \\ +F^2G\alpha & +FG^2\alpha & +G^3\alpha \end{array}\right|)\!(\phi_{01}, -\phi_{10})^2.$$

The various indices of these quantities, being the powers of V by which they must be divided to become absolute invariants, are :—

Index 2, w_2, w'_2, $\mathrm{I}(w_2, w'_2)$;

Index 3, $\mathrm{J}(w_2, w'_2)$, w_3, $\mathfrak{a}_1$;

Index 4, w''_2, $\mathrm{I}(w_2, w''_2)$, $\mathrm{J}(w_2, w_3)$, $\mathfrak{a}_2$, w_4, $\mathfrak{h}_1$;

Index 5, $\mathrm{J}(w_2, w''_2)$, $\mathrm{J}(w_2, w_4)$, $\mathfrak{h}_2$;

Index 6, $\mathfrak{h}_3$;

Index 7, w'_3, $\mathfrak{e}_1$;

Index 8, $\mathrm{J}(w_2, w'_3)$, $\mathfrak{e}_2$.

54. It will be seen from these forms that all the invariants retained are linear in all the quantities L, M, N, P, Q, R, S, α, β, γ, δ, ϵ, a, b, c, k, l, m, n which occur in them. This property facilitates the expression of any other invariant in terms of the various members; thus

$$\frac{\mathrm{LN}-\mathrm{M}^2}{\mathrm{V}^2}=\frac{w_2 w'_2 \mathrm{I}(w_2, w'_2)-\mathrm{J}^2(w_2, w'_2)}{\mathrm{V}^2 w_2^{\,2}},$$

$$\frac{\alpha\epsilon-4\beta\delta+3\gamma^2}{\mathrm{V}^4}=\frac{w_4\mathfrak{h}_3-4\mathrm{J}(w_2, w_4)\,\mathfrak{h}_2+3w_2\mathfrak{h}_1^{\,2}}{w_2^{\,3}},$$

and so for others. Moreover, in the invariants which contain a, b, c linearly, the effect is that the derivatives of ϕ of the second order (being the highest that occurs) are contained linearly; and in those invariants which contain k, l, m, n linearly, the effect is that the derivatives of ϕ of the third order (being the highest order that occurs) are contained linearly, as well as those of the second order.

Moreover, the forms can be used to obtain the value of any given invariant; all that is necessary for this purpose is to obtain the expression of the invariant in terms of the members of the selected aggregate, and to substitute the values of the members that occur. Thus, consider the simultaneous invariant

$$\begin{vmatrix} a, & b, & c \\ \mathrm{L}, & \mathrm{M}, & \mathrm{N} \\ \mathrm{E}, & \mathrm{F}, & \mathrm{G} \end{vmatrix};$$

when expressed in terms of the members of the aggregate, it is equal to

$$\frac{1}{w_2}\{\mathrm{J}(w_2, w''_2)\,\mathrm{I}(w_2, w'_2)-\mathrm{J}(w_2, w'_2)\,\mathrm{I}(w_2, w''_2)\}$$
$$+\frac{1}{w_2^{\,2}}\{w''_2\,\mathrm{J}(w_2, w'_2)-w'_2\,\mathrm{J}(w_2, w''_2)\},$$

and the value of the latter expression is

$$2V^5\left\{\left(H - \frac{1}{\rho'}\right)\frac{dB}{ds} + \frac{1}{\tau'}\frac{dB}{dn}\right\}.$$

In this way the actual values of a large number of the invariants belonging to the asyzygetic aggregate can be obtained. The asyzygetic aggregate of two cubics is known. The asyzygetic aggregate, arising when a quadratic is associated with a system asyzygetically complete in itself, is also known; so that the asyzygetic aggregate belonging to w_2, w'_2, w''_2, w_3, w'_3 can be obtained by the application of known theorems.

Further, the asyzygetic aggregate of a cubic and a quartic is known, so that the asyzygetic aggregate could be obtained for w_2, w'_2, w''_2, w_3, w_4, and also for w_2, w'_2, w''_2, w'_3, w_4. But, so far as I am aware, the asyzygetic aggregate of either two cubics and one quartic, or a cubic and any system asyzygetically complete in itself, is not known; as soon as either is known, the results could be applied to obtain the asyzygetic aggregate for w_2, w'_2, w''_2, w_3, w'_3, w_4, that is, the complete system of concomitants in terms of which any rational integral invariant can be expressed as a rational integral function.

The Geometrical Magnitudes which are Independent.

55. As regards the quantities, which have served to assign the geometrical significance of the several invariants, some inferences can be drawn from the results obtained. Denoting by χ the angle between the curve and the line of curvature connected with ρ_1, we have

$$\frac{1}{\rho'} = \frac{\cos^2\chi}{\rho_1} + \frac{\sin^2\chi}{\rho_2},$$

$$\frac{1}{\tau'} = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\cos\chi\sin\chi,$$

$$H = \frac{1}{\rho_1} + \frac{1}{\rho_2}, \quad K = \frac{1}{\rho_1\rho_2},$$

so that not more than three of the quantities $\frac{1}{\rho'}$, $\frac{1}{\tau'}$, H, K are independent. For purposes of expression, we have retained $\frac{1}{\rho'}$, $\frac{1}{\tau'}$, H. There are also the quantities B and $\frac{1}{\rho''}$.

To the order of derivatives which occur in the invariants that have been constructed, the geometric magnitudes, which might be expected to occur in the values of the invariants, are as follows:—

$$\frac{1}{\rho'},\ \frac{d}{ds}\left(\frac{1}{\rho'}\right),\ \frac{d}{dn}\left(\frac{1}{\rho'}\right),\ \frac{d^2}{ds^2}\left(\frac{1}{\rho'}\right),\ \frac{d^2}{ds\,dn}\left(\frac{1}{\rho'}\right),\ \frac{d^2}{dn\,ds}\left(\frac{1}{\rho'}\right),\ \frac{d^2}{dn^2}\left(\frac{1}{\rho'}\right),$$

$$\mathrm{H},\ \frac{d\mathrm{H}}{ds},\ \frac{d\mathrm{H}}{dn},\ \frac{d^2\mathrm{H}}{ds^2},\ \frac{d^2\mathrm{H}}{ds\,dn},\ \frac{d^2\mathrm{H}}{dn\,ds},\ \frac{d^2\mathrm{H}}{dn^2},$$

$$\mathrm{B},\ \frac{d\mathrm{B}}{ds},\ \frac{d\mathrm{B}}{dn},\ \frac{d^2\mathrm{B}}{ds^2},\ \frac{d^2\mathrm{B}}{ds\,dn},\ \frac{d^2\mathrm{B}}{dn\,ds},\ \frac{d^2\mathrm{B}}{dn^2},$$

$$\frac{1}{\rho''},\ \frac{d}{ds}\left(\frac{1}{\rho''}\right),\ \frac{d}{dn}\left(\frac{1}{\rho''}\right),$$

$$\frac{1}{\tau'},$$

and the derivatives of $\frac{1}{\tau'}$. But not all of these can be retained as independent magnitudes. In § 40 it was proved that

$$\frac{d}{ds}\left(\frac{1}{\tau'}\right) = \frac{2}{\tau'\mathrm{B}}\frac{d\mathrm{B}}{ds} - \left(\mathrm{H} - \frac{2}{\rho'}\right)\frac{1}{\rho''} - \frac{d}{dn}\left(\frac{1}{\rho'}\right),$$

$$\frac{d}{dn}\left(\frac{1}{\tau'}\right) = \left(\mathrm{H} - \frac{2}{\rho'}\right)\frac{1}{\mathrm{B}}\frac{d\mathrm{B}}{ds} + \frac{d}{ds}\left(\frac{1}{\rho'}\right) + \frac{2}{\rho''\tau'} - \frac{d\mathrm{H}}{ds};$$

so that the first derivatives of $\frac{1}{\tau'}$, and consequently also the second (and higher) derivatives, are expressible in terms of the derivatives of the other quantities retained.* Again, in §§ 41, 43, 47 it has been shown that the quantities

$$\frac{d^2}{ds\,dn}\left(\frac{1}{\rho'}\right) - \frac{d^2}{dn\,ds}\left(\frac{1}{\rho'}\right),$$

$$\frac{d^2\mathrm{H}}{ds\,dn} - \frac{d^2\mathrm{H}}{dn\,ds},$$

$$\frac{d^2\mathrm{B}}{ds\,dn} - \frac{d^2\mathrm{B}}{dn\,ds},$$

are expressible in terms of the derivatives of the first order; so that it is sufficient to retain $\frac{d^2}{ds\,dn}\left(\frac{1}{\rho'}\right)$, $\frac{d^2\mathrm{H}}{ds\,dn}$, $\frac{d^2\mathrm{B}}{ds\,dn}$, and reject $\frac{d^2}{dn\,ds}\left(\frac{1}{\rho'}\right)$, $\frac{d^2\mathrm{H}}{dn\,ds}$, $\frac{d^2\mathrm{B}}{dn\,ds}$. Further, in § 41, it was proved that

* It is proved in DARBOUX's 'Théorie générale des Surfaces,' vol. 2, p. 360, that the quantity

$$\frac{d}{ds}\left(\frac{1}{\tau'}\right) + \left(\mathrm{H} - \frac{2}{\rho'}\right)\frac{1}{\rho''},$$

which occurs in the first of the two equations, is the same for two curves that have the same tangent.

$$\frac{1}{B}\frac{d^2B}{ds^2} = 2K - \frac{d}{dn}\left(\frac{1}{\rho''}\right) + \frac{1}{\rho''^2} + 2\left(\frac{1}{B}\frac{dB}{ds}\right)^2,$$

$$\frac{d^2}{dn^2}\left(\frac{1}{\rho'}\right) = \frac{d^2H}{ds^2} - K\left(H - \frac{2}{\rho''}\right) - \frac{1}{\rho''}\frac{dH}{dn} - \frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right)$$
$$+ \frac{1}{B}\left\{\frac{2}{\tau'}\frac{d^2B}{dn\,ds} - 4\frac{dB}{ds}\frac{dH}{ds} + 5\frac{dB}{ds}\frac{d}{dt}\left(\frac{1}{\rho'}\right)\right\}$$
$$+ \frac{2}{B^2}\frac{dB}{ds}\left\{\left(H - \frac{2}{\rho'}\right)\frac{dB}{ds} - \frac{1}{\tau'}\frac{dB}{dn}\right\},$$

and the values of $\frac{d}{dt}\left(\frac{1}{\rho'}\right)$ and $\frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right)$ have been given in §§ 36, 41; hence it is unnecessary to retain $\frac{d^2B}{ds^2}$, $\frac{d^2}{dn^2}\left(\frac{1}{\rho'}\right)$.

We therefore retain the quantities

$$\frac{1}{\rho'},\ \frac{d}{ds}\left(\frac{1}{\rho'}\right),\ \frac{d}{dn}\left(\frac{1}{\rho'}\right),\ \frac{d^2}{ds^2}\left(\frac{1}{\rho'}\right),\ \frac{d^2}{ds\,dn}\left(\frac{1}{\rho'}\right),$$

$$H,\ \frac{dH}{ds},\ \frac{dH}{dn},\ \frac{d^2H}{ds^2},\ \frac{d^2H}{ds\,dn},\ \frac{d^2H}{dn^2},$$

$$B,\ \frac{dB}{ds},\ \frac{dB}{dn},\ \frac{d^2B}{ds\,dn},\ \frac{d^2B}{dn^2},$$

$$\frac{1}{\rho''},\ \frac{d}{ds}\left(\frac{1}{\rho''}\right),\ \frac{d}{dn}\left(\frac{1}{\rho''}\right),$$

$$\frac{1}{\tau'},$$

being 20 in all; their association with the 20 algebraically independent differential invariants set out in § 53 has already been made.

56. These results would seem to have an important bearing when we proceed to the next higher order of derivatives. As $\frac{d^2B}{ds^2}$ is rejected from the aggregate of quantities, the quantities $\frac{d^3B}{ds^3}$ and $\frac{d^3B}{dn\,ds^2}$ will also be rejected; also, as $\frac{d^2B}{dn\,ds} - \frac{d^2B}{ds\,dn}$ is expressible by quantities of lower order, the quantities $\frac{d^3B}{ds^2dn}$ and $\frac{d^3B}{dn\,ds\,dn}$ will be rejected; thus, in this order, the only derivatives of B to be retained are

$$\frac{d^3B}{ds\,dn\,ds},\quad \frac{d^3B}{dn^2\,ds},\quad \frac{d^3B}{ds\,dn^2},\quad \frac{d^3B}{dn^3},$$

four in number. Moreover, these four may reduce to two; for the first may be equivalent to the rejected $\frac{d^3\mathrm{B}}{dn\,ds^2}$, while the second and the third may be equivalent to one another. Similarly, the only derivatives of $\frac{1}{\rho'}$ to be retained are

$$\frac{d^3}{ds^3}\left(\frac{1}{\rho'}\right),\quad \frac{d^3}{ds^2\,dn}\left(\frac{1}{\rho'}\right),\quad \frac{d^3}{dn\,ds\,dn}\left(\frac{1}{\rho'}\right),\quad \frac{d^3}{dn\,ds^2}\left(\frac{1}{\rho'}\right),$$

four in number; and these may reduce to two. There are six derivatives of H, viz.:

$$\frac{d^3\mathrm{H}}{ds^3},\quad \frac{d^3\mathrm{H}}{ds^2\,dn},\quad \frac{d^3\mathrm{H}}{dn\,ds^2},\quad \frac{d^3\mathrm{H}}{dn^2\,ds},\quad \frac{d^3\mathrm{H}}{ds\,dn^2},\quad \frac{d^3\mathrm{H}}{dn^3},$$

which may reduce to four; and there are four derivatives of $\frac{1}{\rho''}$, viz.:

$$\frac{d^2}{ds^2}\left(\frac{1}{\rho''}\right),\quad \frac{d^2}{ds\,dn}\left(\frac{1}{\rho''}\right),\quad \frac{d^2}{dn\,ds}\left(\frac{1}{\rho''}\right),\quad \frac{d^2}{dn^2}\left(\frac{1}{\rho''}\right),$$

which may reduce to three. Hence there are, in all, eighteen new geometrical quantities arising through the inclusion of derivatives of the next higher order; and these eighteen quantities may reduce to eleven.

Now the number of differential invariants, which involve derivatives of ϕ up to order n and the corresponding quantities of proper order, is $\frac{1}{2}n(3n+5)$ by § 27; and this number is certainly subject to diminution by 1 unit, as explained at the beginning of § 29, so that it is $\frac{1}{2}n(3n+5)-1$. When $n=4$, this is 33; and we know that there are 20 invariants for $n=3$; so that 13 new invariants are introduced by the differential equations for the new order. It has been indicated that there may be only 11 new geometrical quantities available for their expression; if so, the inference would be that there are two algebraic relations among these 13. These relations are outside the differential equations; and the only cause from which they could arise would be owing to the intrinsic significance of the magnitudes. As there actually is one* differential invariant of deformation of this order (that is, a function involving E, F, G and their derivatives up to the third order, and no other quantities), the obvious suggestion is that it would behave like the invariant of the lower order, due to Gauss, and would be expressible in terms of invariants in the binariant system composed of the fundamental magnitudes; but this inference is only a suggestion, and cannot be regarded as an established result.

[Note: *added*, 12 *May*, 1903.

After the manuscript of this memoir had been sent to the Royal Society but before the memoir itself had been read, I succeeded in definitely establishing the inference suggested at the end of § 56. The necessary calculations are long and are of the

* Żorawski, *l.c.*, p. 31.

same general character as those in §§ 14–22; their aim is to obtain the one solution, other than ∇ and V^2, of the twenty-eight partial differential equations satisfied by differential invariants of deformation, which are of order not higher than three. The mode of dealing with such a system of equations has been amply illustrated in Part I. of the memoir; accordingly, only the results of the analysis will be given.

We denote by Γ, Γ', Γ'', Δ, Δ', Δ'', quantities defined in § 6; and we write

$$u = E_{12} - 2F_{21} + G_{30} + \Gamma''E_{20} - (2\Gamma' + \Delta'')E_{11} + \Gamma E_{02} + 2\Delta''F_{20} - 2\Delta'G_{20} + \Delta G_{11},$$

$$v = E_{03} - 2F_{12} + G_{21} + \Gamma''E_{11} - 2\Gamma'E_{02} + 2\Gamma F_{02}, + \Delta''G_{20} - (\Gamma + 2\Delta')G_{11} + \Delta G_{02},$$

$$\theta = E_{02} - 2F_{11} + G_{20},$$

these being simultaneous solutions of the eighteen equations, which correspond to the vanishing of the derivatives of ξ and η of order 4 and of order 3 in the various arguments (§ 13). Further, we write

$$\begin{aligned} p = {} & E^2(-4E_{01}G_{10}G_{01} + 8F_{10}G_{10}G_{01} - 4G_{10}{}^3) \\ & + EF(-4E_{10}G_{10}G_{01} + 8E_{01}F_{10}G_{01} - 16F_{10}F_{01}G_{10} - 16F_{10}{}^2G_{01} + 16F_{10}G_{10}{}^2 \\ & \qquad + 4E_{01}G_{10}{}^2 + 8E_{01}F_{01}G_{10}) \\ & + EG(-2E_{10}E_{01}G_{01} + 6E_{10}F_{10}G_{01} + 4E_{10}F_{01}G_{10} - 4E_{10}G_{10}{}^2 - 4E_{01}F_{10}F_{01} \\ & \qquad - 2E_{01}F_{10}G_{10} + 8F_{01}F_{10}{}^2 - 4F_{10}{}^2G_{10} - 2E_{01}{}^2G_{10}) \\ & + F^2(-2E_{10}E_{01}G_{01} + 4E_{10}F_{01}G_{10} + 10E_{10}F_{10}G_{01} - 4E_{10}G_{10}{}^2 - 12F_{10}{}^2G_{10} \\ & \qquad + 24F_{10}{}^2F_{01} - 6E_{01}F_{10}G_{10} - 12E_{01}F_{10}F_{01} - 2E_{01}{}^2G_{10}) \\ & + FG(8E_{10}E_{01}F_{01} - 4E_{10}{}^2G_{01} + 4E_{10}E_{01}G_{10} - 32E_{10}F_{10}F_{01} + 16E_{10}F_{10}G_{10} + 8E_{01}{}^2F_{10}) \\ & + G^2(8E_{10}{}^2F_{01} - 4E_{10}{}^2G_{10} - 4E_{01}{}^2E_{10}), \end{aligned}$$

$$\begin{aligned} q = {} & E^2(8F_{10}G_{01}{}^2 - 4E_{01}G_{01}{}^2 - 4G_{01}G_{10}{}^2) \\ & + EF(-4E_{10}G_{01}{}^2 - 32F_{10}F_{01}G_{01} + 8F_{10}G_{10}G_{01} + 4E_{01}G_{10}G_{01} + 16E_{01}F_{01}G_{01} \\ & \qquad + 8F_{01}G_{10}{}^2) \\ & + EG(6E_{10}F_{01}G_{01} - 2E_{10}G_{10}G_{01} + 4E_{01}F_{10}G_{01} + 8F_{10}F_{01}{}^2 - 4F_{10}F_{01}G_{10} \\ & \qquad - 4E_{01}{}^2G_{01} - 2E_{01}F_{01}G_{10} - 4E_{01}F_{01}{}^2 - 2E_{01}G_{10}{}^2) \\ & + F^2(10E_{01}F_{01}G_{01} - 2E_{10}G_{10}G_{01} + 4E_{01}F_{10}G_{01} + 24F_{01}{}^2F_{10} - 12F_{10}F_{01}G_{10} \\ & \qquad - 2E_{01}G_{10}{}^2 - 6E_{01}F_{01}G_{10} - 4E_{01}{}^2G_{01} - 12E_{01}F_{01}{}^2) \\ & + FG(-4E_{10}E_{01}G_{01} - 16E_{10}F_{01}{}^2 + 8E_{10}F_{01}G_{10} - 16E_{01}F_{10}F_{01} + 8E_{01}F_{10}G_{10} \\ & \qquad + 16E_{01}{}^2F_{01} + 4E_{01}{}^2G_{10}) \\ & + G^2(8E_{10}E_{01}F_{01} - 4E_{10}E_{01}G_{10} - 4E_{01}{}^3). \end{aligned}$$

Also, we write

$$\lambda_1 = 4V^4u - 4V^4\theta(2\Delta' + 3\Gamma) - p,$$

$$\lambda_2 = 4V^4v - 4V^4\theta(2\Gamma' + 3\Delta'') - q.$$

Then *a first expression for the differential invariant of deformation of the third order* is found to be

$$(E\lambda_2^2 - 2F\lambda_1\lambda_2 + G\lambda_1^2)V^{-14}.$$

This expression can be modified by means of the relation (§ 35)

$$\begin{aligned}4V^2(LN - M^2) &= \nabla \\ &= -2V^2\theta + E\{(E_{01} - 2F_{10})G_{01} + G_{10}^2\} + G\{E_{01}^2 - E_{10}(2F_{01} - G_{10})\} \\ &\quad + F\{E_{10}G_{01} - E_{01}(2F_{01} + G_{10}) + 2F_{10}(2F_{01} - G_{10})\}.\end{aligned}$$

Dividing both sides by V^2 and taking the derivative with regard to x, we find (on using the relations in § 6, and after reduction) that we have

$$\lambda_1 = -8V^4(NP - 2MQ + LR), = -8V^4\mathfrak{a},$$

say. Proceeding similarly from the derivative with regard to y, we have

$$\lambda_2 = -8V^4(NQ - 2MR + LS), = -8V^4\mathfrak{b},$$

say. It thus appears that *the two combinations of* E, F, G *and their derivatives up to the third order, represented by* λ_1 *and* λ_2, *are expressible in terms of the fundamental magnitudes of the second and the third order.* Moreover, dropping the numerical factor 64, we have *an expression for the differential invariant of deformation of the third order* (say I) in the form

$$IV^6 = E\mathfrak{b}^2 - 2F\mathfrak{a}\mathfrak{b} + G\mathfrak{a}^2.$$

By the theory in the preceding memoir, this invariant (which now involves only fundamental magnitudes of the first three orders and none of their derivatives) ought to be expressible in terms of the members of the system set out in § 53. Writing

$$w'_1 = (\mathfrak{a}, \mathfrak{b})(\phi_{01}, -\phi_{10}),$$

$$w''_1 = (E\mathfrak{b} - F\mathfrak{a}, F\mathfrak{b} - G\mathfrak{a})(\phi_{01}, -\phi_{10}),$$

we find

$$w_2V^6I = V^2w'^2_1 + w''^2_1,$$

$$w_2^2w'_1 = w_2w_3I(w_2, w'_2) + w_2w'_2\mathfrak{a}_1 - 2J(w_2, w'_2)J(w_2, w_3) - 2V^2w'_2w_3,$$

$$\begin{aligned}w_2^2w''_1 = {} & w_2I(w_2, w'_2)J(w_2, w_3) + 2V^2w_3J(w_2, w'_2) - 2V^2w'_2J(w_2, w_3) \\ & - 2w_2\mathfrak{a}_1J(w_2, w'_2) + w_2w'_2\mathfrak{a}_2.\end{aligned}$$

When the geometric values of the several invariants are substituted, we find

$$w'_1 = \mathrm{BV}^3 \frac{d\mathrm{K}}{ds},$$

$$w''_1 = \mathrm{BV}^4 \frac{d\mathrm{K}}{dn};$$

and therefore

$$\mathrm{I} = \left(\frac{d\mathrm{K}}{ds}\right)^2 + \left(\frac{d\mathrm{K}}{dn}\right)^2,$$

which is *the geometric significance of the differential invariant of deformation of the third order.* Its expression appears to involve association with the curve $\phi = 0$; but the relations in § 51 shew that the association is the same for all curves, so that the quantity is a function solely of position on the surface, being the sum of the squares of the first derivatives of K along any two perpendicular directions along the surface.]

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